Extending the Fundamental Theorem of Linear Programming for Strict Inequalities

ISSAC '21

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July 22, 2021

hybr i2 d Theory of Hybrid Systems Informatik 2 | RWTH AACHEN UNIVERSITY

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- King gave a proof in his dissertation in 2014 specifically for this algorithm
- our formulation of the construction is generally applicable to other algorithms
- our constructive proof gives insight into the problem

Linear real arithmetic (LRA)

$$\varphi = (x_1 + x_2 \leq 3 \vee x_1 - x_2 \geq 4) \wedge (x_1 - x_2 \geq 2 \vee x_1 + x_2 \geq 5) \wedge x_2 \geq 1$$

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$$\exists \alpha : \{\boxed{x_1}, \ldots, \boxed{x_n}\} \to \mathcal{U}.\ \alpha \models \varphi\ ?$$

Lazy SMT solving

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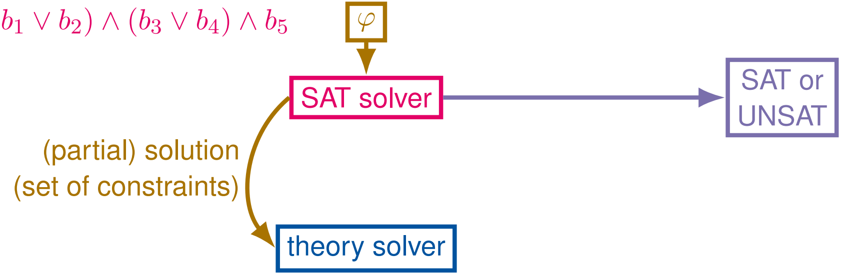
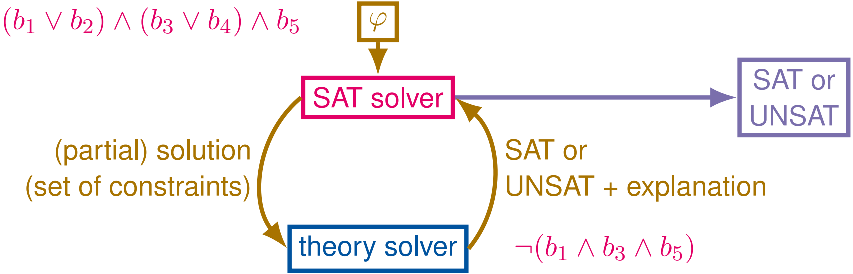


$x_1 + x_2 \leq 3 \;\wedge\; x_1 - x_2 \geq 2 \;\wedge x_1 + x_2 < 5 \wedge\; x_2 \geq 1$

Lazy SMT solving



$$\boxed{x_1 + x_2 \leq 3} \wedge \boxed{x_1 - x_2 \geq 2} \wedge x_1 + x_2 < 5 \wedge \boxed{x_2 \geq 1}$$

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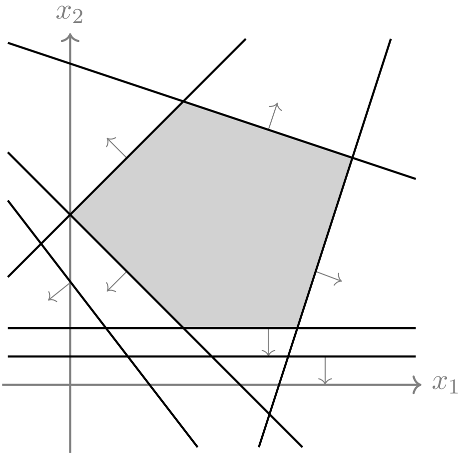
- Linear program

$$\max_{\boldsymbol{x} \in \mathcal{U}^n} \boldsymbol{c}^T \cdot \boldsymbol{x}$$

subject to $A\boldsymbol{x} \leq \boldsymbol{b}$ (denoted by C)

where $\boldsymbol{c} \in \mathcal{F}^n, A \in \mathcal{F}^{m \times n}, \boldsymbol{b} \in \mathcal{U}^m$

Geometric interpretation



Assuming C has full rank

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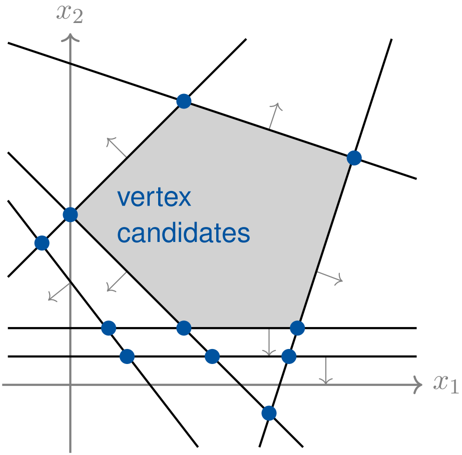

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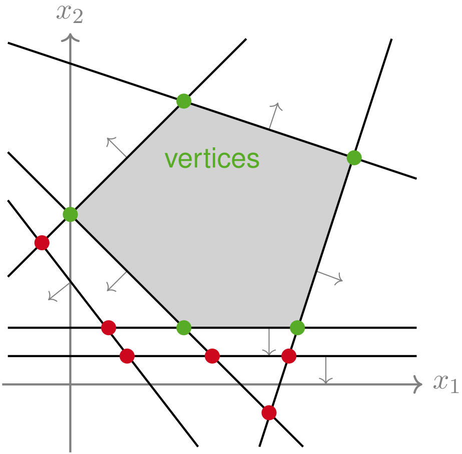
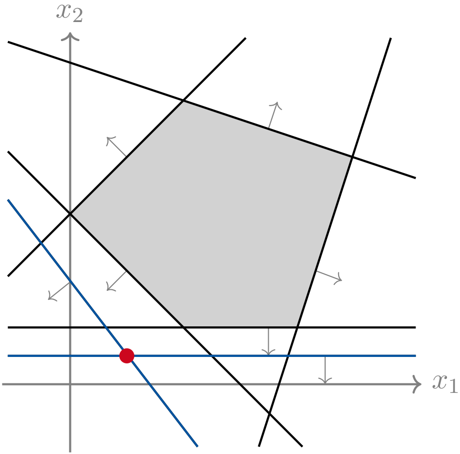


"C is satisfiable iff it has a satisfying vertex candidate"

Simplex for lazy SMT [Dutertre et al 2006]



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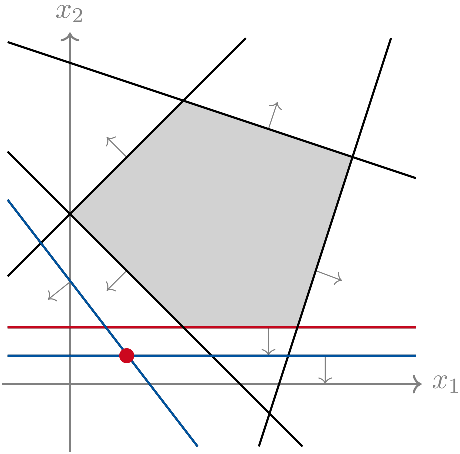

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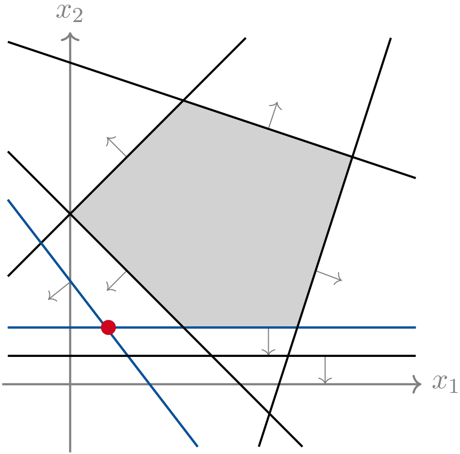

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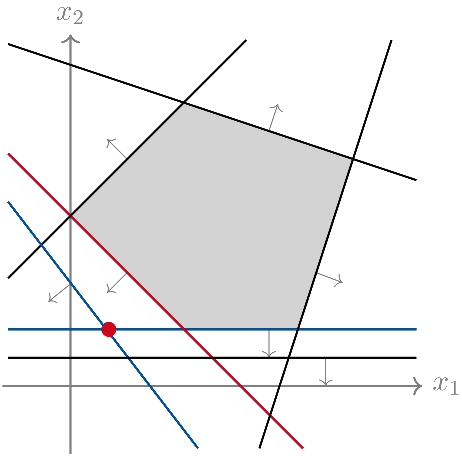

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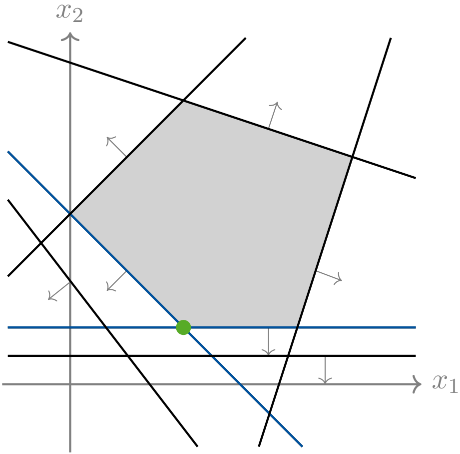
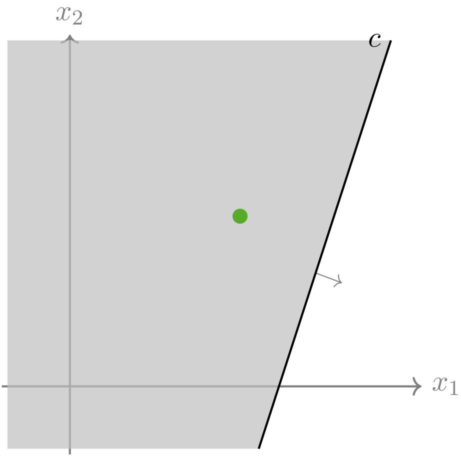


How to prove the
fundamental theorem of linear programming?

Tight constraints

$\alpha \models \underbrace{p \leq b}_{c}$



Tight constraints

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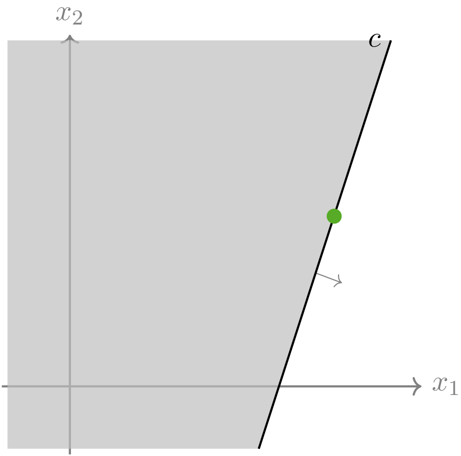
c is tight under α iff

$$\alpha \models \underbrace{p = b}_{\tilde{c}}$$



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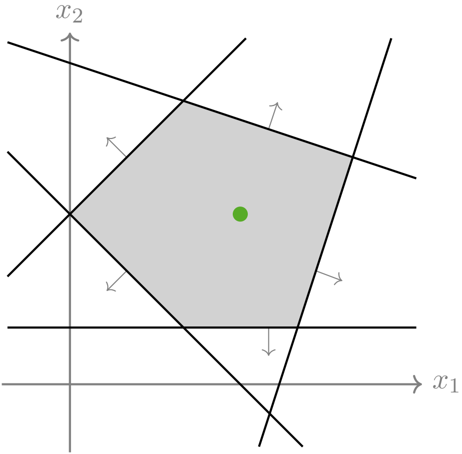

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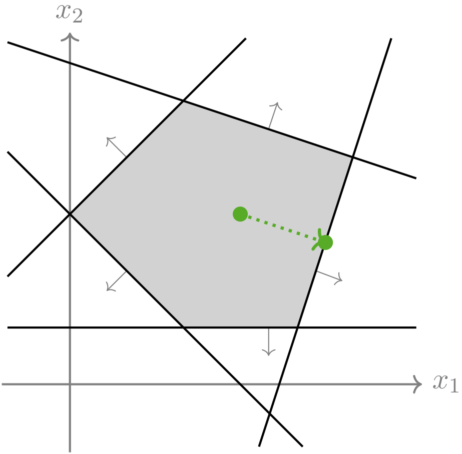

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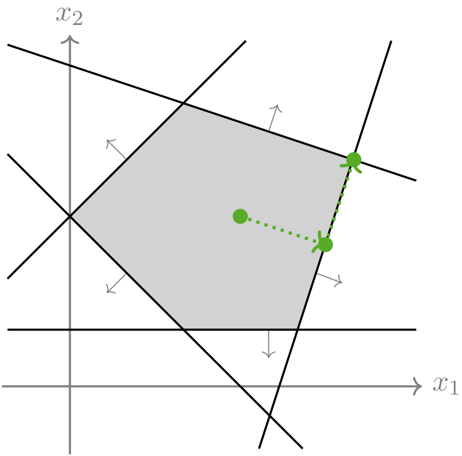

Linear independence

- system of linear constraints

$$C = \{\overbrace{\underbrace{a_{i,1} \cdot x_1 + \ldots + a_{i,n} \cdot x_n}_{p_i} \sim_i b_i}^{c_i} \mid i = 1, \ldots, m\}$$

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- $\{p_1, \ldots, p_k\}$ linearly independent iff

$$f_1 \cdot p_1 + \ldots + f_k \cdot p_k = 0 \iff f_1 = \ldots = f_k = 0$$

for all $f_1, \ldots, f_m \in \mathcal{F}$

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  for all $f_1, \ldots, f_m \in \mathcal{F}$
- $V = \{c_1, \ldots, c_k\}$ linearly independent iff $\{p_1, \ldots, p_k\}$ linearly independent

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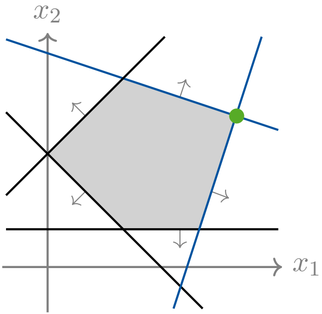
  - define set MLI_C of such $V \subseteq C$

Fundamental theorem of linear programming

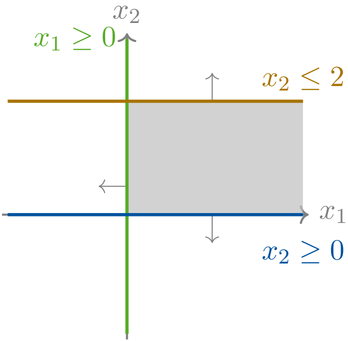
- C is satisfiable iff $\exists$ maximal linearly independent $V \subseteq C$ such that

$$\exists \alpha : X \to \mathcal{U}.\ \alpha \models \tilde{V} \cup C.$$



Strict constraints

So far, only weak relations $\leq, \geq, =$ are considered



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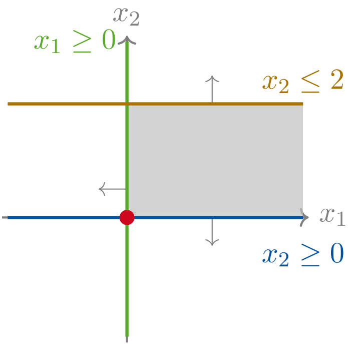

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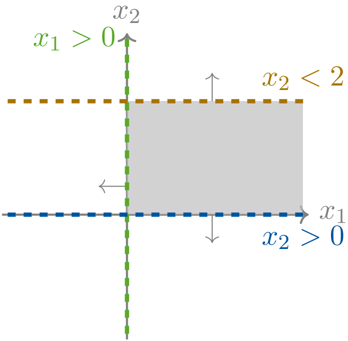

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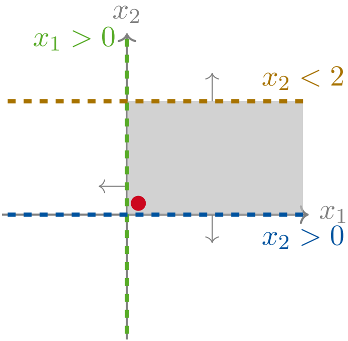

Weakened systems

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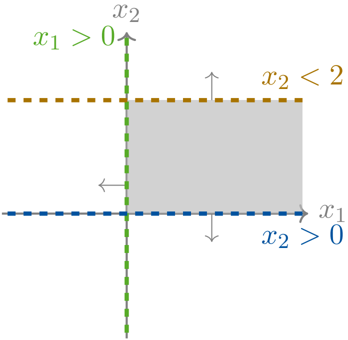
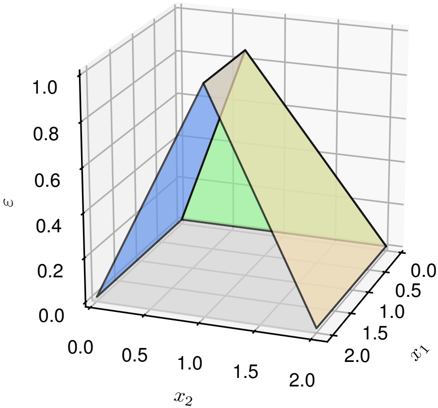
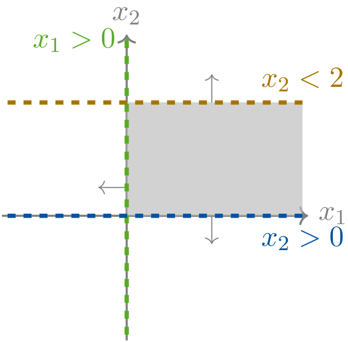
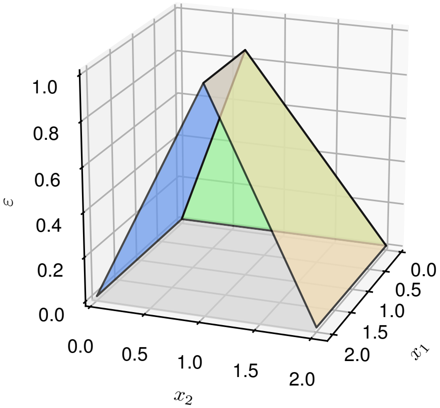




Weakened systems

$$\exists \alpha \models C \iff \exists \alpha \models C_w \wedge \varepsilon > 0$$





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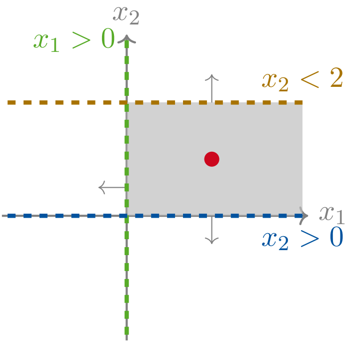

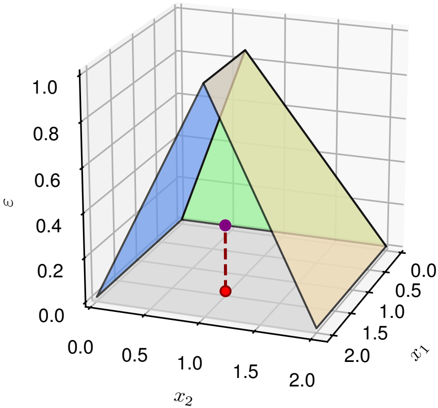

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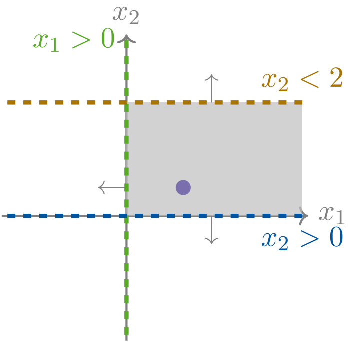
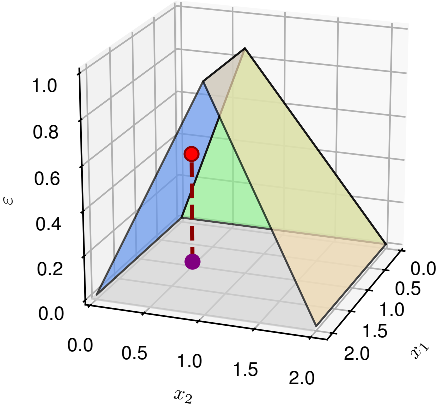
$$\alpha \models C \implies \exists g > 0.\ \forall e \in (0, g].\ \alpha[\varepsilon \mapsto e] \models C_w.$$

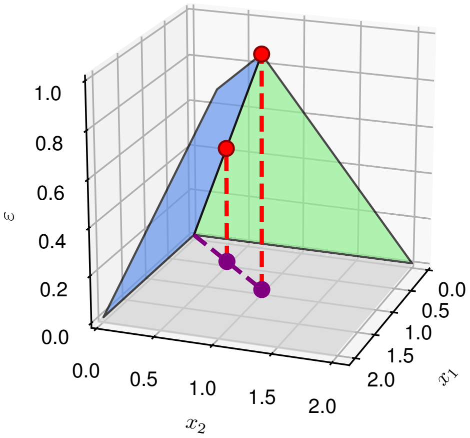




Application of the fundamental theorem of LP

$$\exists \alpha.\ \alpha \models C$$

$$\implies \exists g \in \mathcal{U}_{>0}.\ \forall e \in \mathcal{U}_{>0}.\ (e \leq g \rightarrow \exists \alpha : X \rightarrow \mathcal{U}.\ \alpha[\varepsilon \mapsto e] \models \tilde{V_w} \cup C_w).$$



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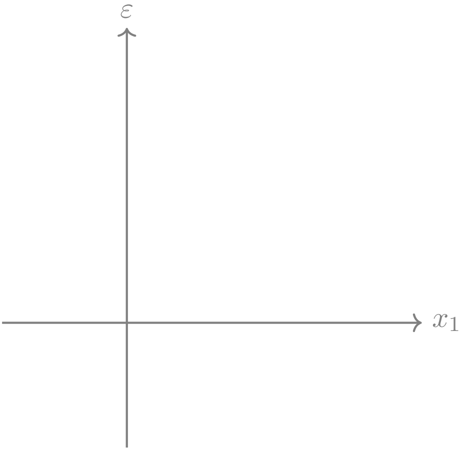
$2 \cdot x_1 > 0 \wedge \frac{1}{2} \cdot x_1 > -2$

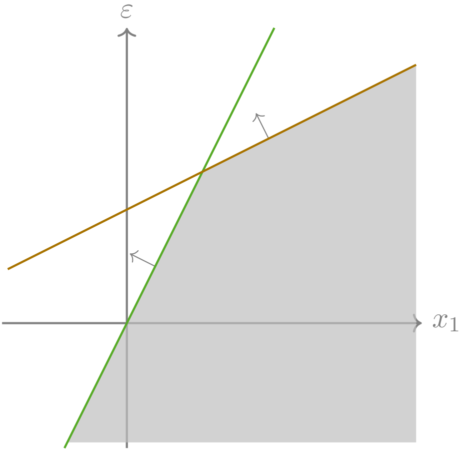


Application of the fundamental theorem of LP

$$2 \cdot x_1 > 0 \land \frac{1}{2} \cdot x_1 > -2$$

transformed to

$$2 \cdot x_1 \geq \varepsilon \land \frac{1}{2} \cdot x_1 \geq -2 + \varepsilon$$



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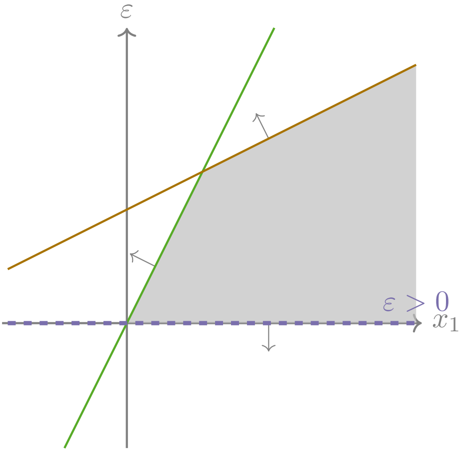

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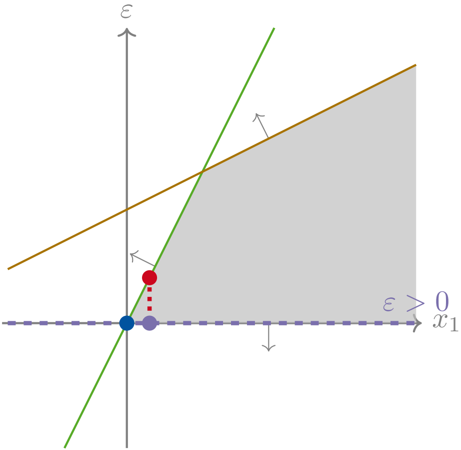

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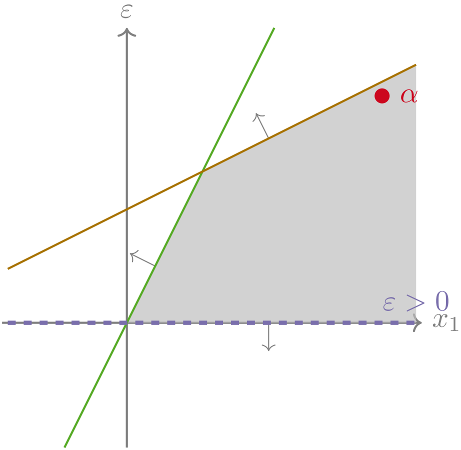

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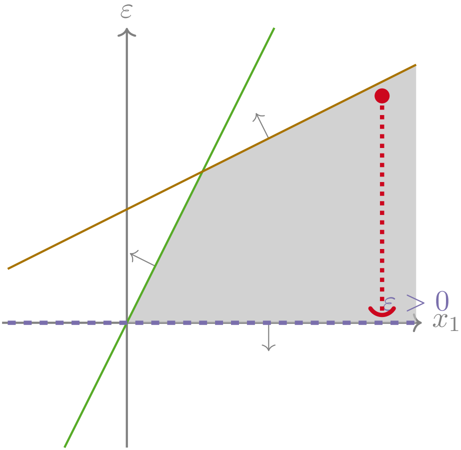

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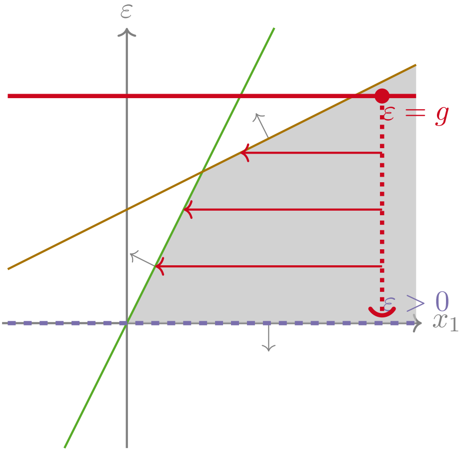

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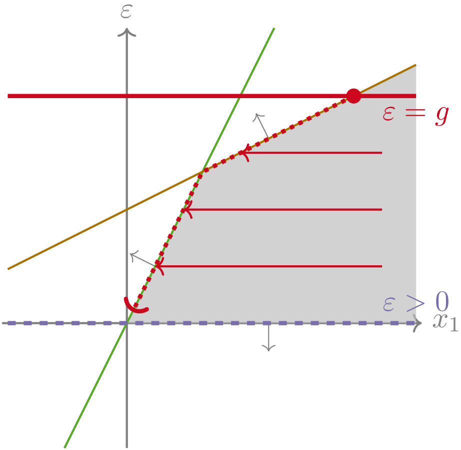

Application of the fundamental theorem of LP

$$\exists \alpha \models$$
$$2 \cdot x_1 > 0 \wedge \frac{1}{2} \cdot x_1 > -2$$

$$\Longleftrightarrow$$

$\exists g > 0.\ \forall e \in (0, g].\ \exists V \in \mathsf{MLI}_C.$
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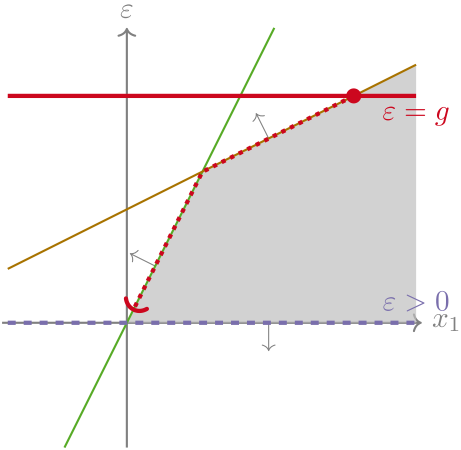

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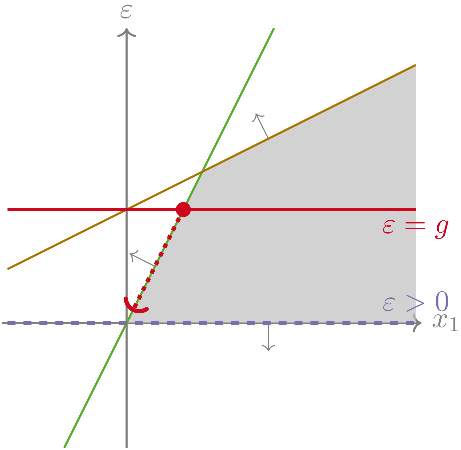
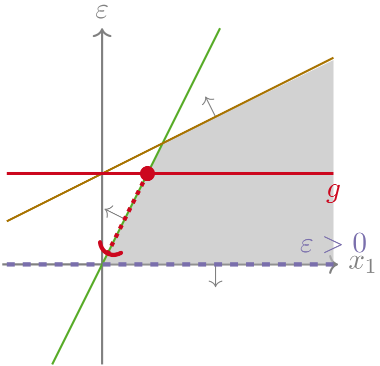


Intermediate result

- C satisfiable iff $\exists$ maximal linearly independent $V \subseteq C$ such that

$$\exists g \in \mathcal{U}_{>0}.\ \forall e \in \mathcal{U}_{>0}.\ (e \leq g \to \exists \alpha : X \to \mathcal{U}.\ \alpha[\varepsilon \mapsto e] \models \tilde{V}_w \cup C_w).$$



Infinitesimal arithmetic

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$$(d_1 + e_1 \cdot \varepsilon) < (d_2 + e_2 \cdot \varepsilon) :\iff d_1 < d_2 \vee (d_1 = d_2 \wedge e_1 < e_2)$$

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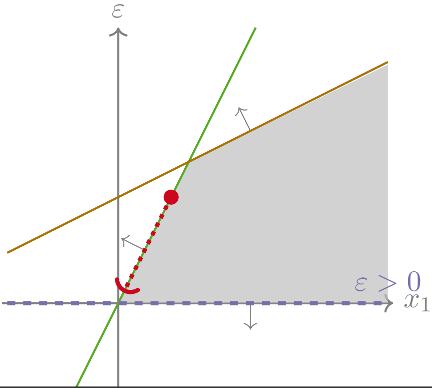
- $\cdot : \mathcal{F} \times \mathcal{U}[\varepsilon] \to \mathcal{U}[\varepsilon]$ with

$$a \cdot (d + e \cdot \varepsilon) \mapsto a \cdot d + a \cdot e \cdot \varepsilon$$

Final theorem

- system C of linear constraints, C_w^* is C_w interpreted over $(\mathcal{U}[\varepsilon], <')$



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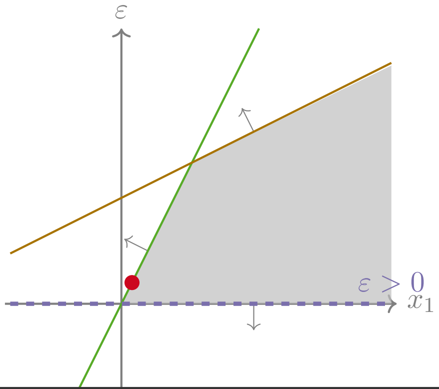

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Reminder

- system C of weak linear constraints
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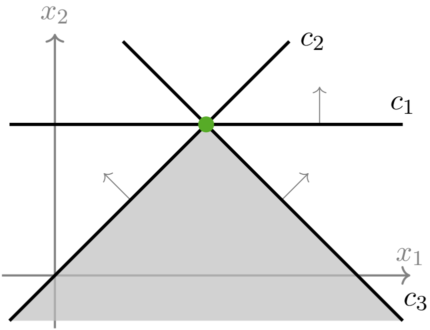
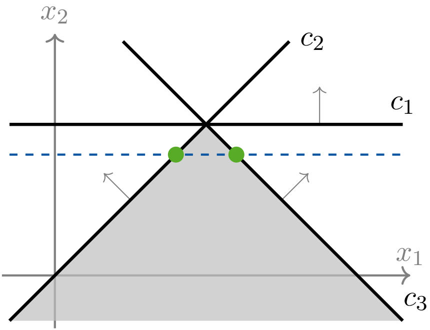
Discussion



$$c_1 : x_2 \leq 2$$
$$c_2 : x_2 - x_1 \leq 0$$
$$c_3 : x_2 + x_1 - 4 \leq 0$$

Discussion



$$c_1 : x_2 \leq 2 - \varepsilon$$

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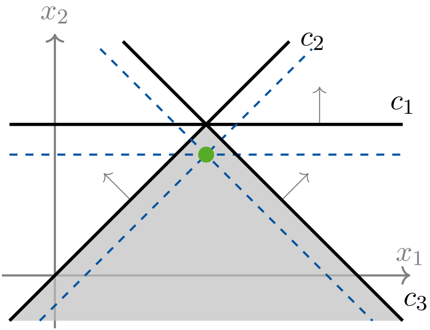
Discussion



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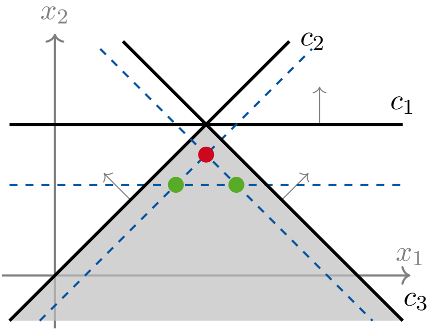

$$c_1 : \frac{1}{2}x_2 \leq 2 - \varepsilon$$

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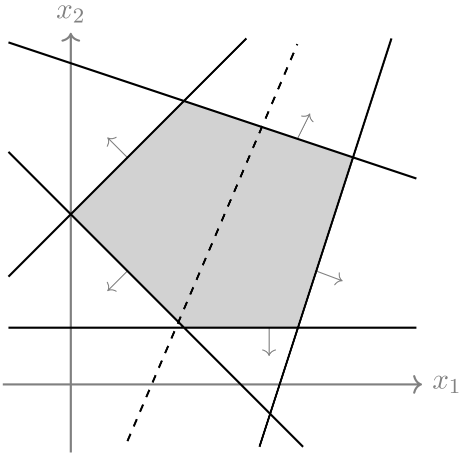
Not-equal

$$(C \wedge p \neq b) \equiv (C \wedge p < b) \vee (C \wedge p > b)$$



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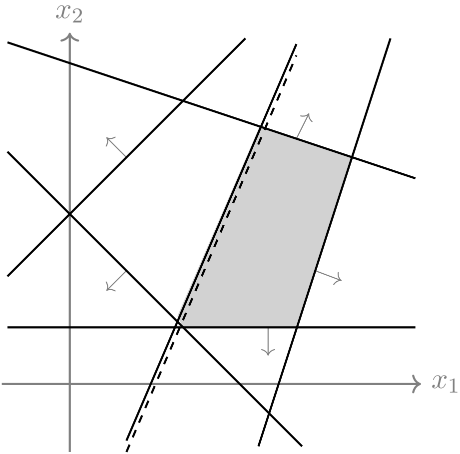

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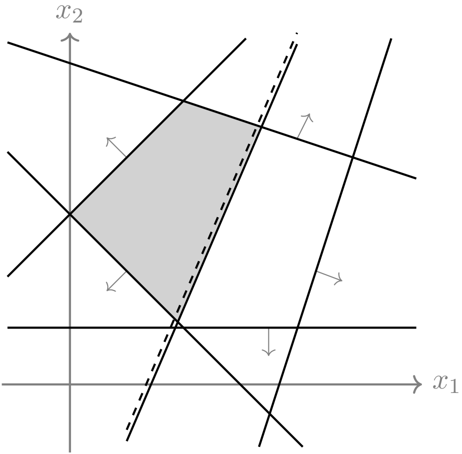

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  - applications outside SMT solving