Levelwise construction of a single cylindrical algebraic cell

Dagstuhl Seminar on 'New Perspectives in Symbolic Computation and Satisfiability Checking' (22072)

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Christopher W. Brown, James H. Davenport, Matthew England

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Introduction

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  - McCallum, Brown-McCallum, Equational constraints, Lazard, ...

Non-linear arithmetic

Is a given Boolean combination of polynomial constraints satisfiable?

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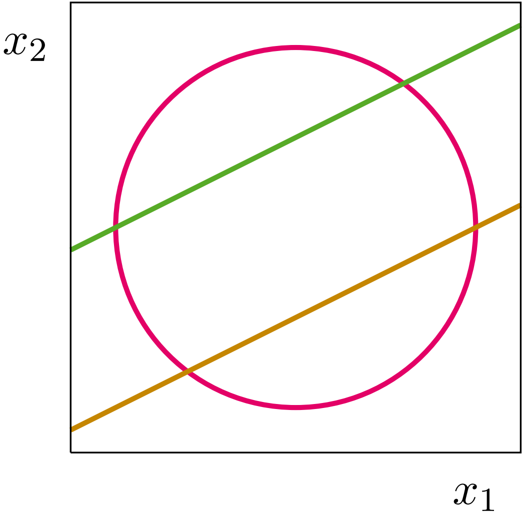
$$0.5x + 0.5 - y > 0 \land x^2 + y^2 - 1 < 0 \land 0.5x - 0.5 - y < 0$$

Cylindrical algebraic decomposition and single cells



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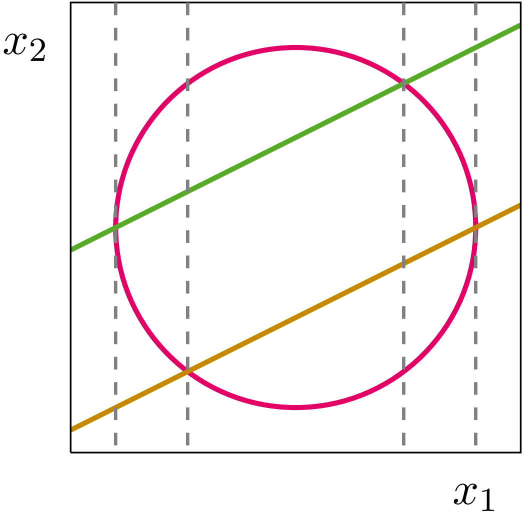

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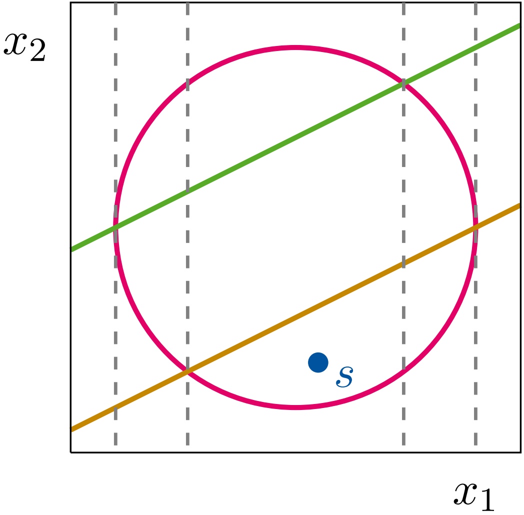

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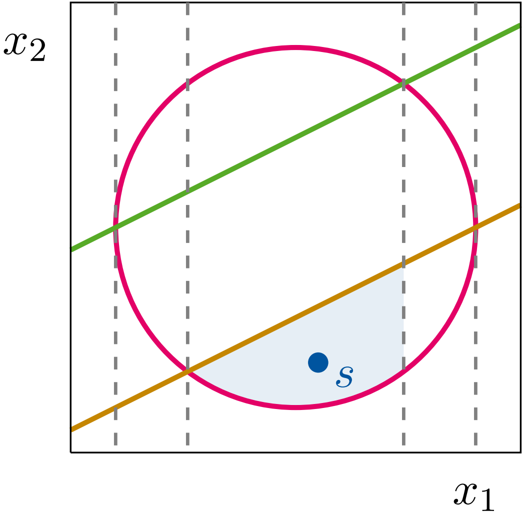

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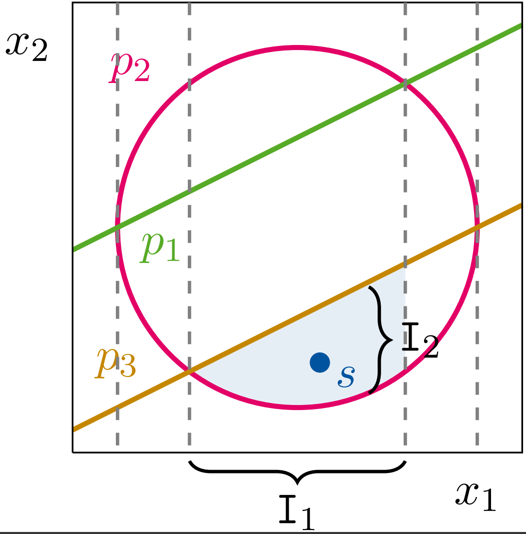

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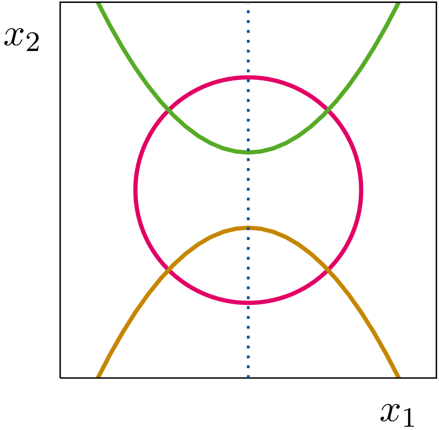
NLSAT explanations and cylindrical algebraic coverings

$$x^2 - y > -0.5 \wedge x^2 + y^2 > 1 \wedge -x^2 - y < 0.5$$



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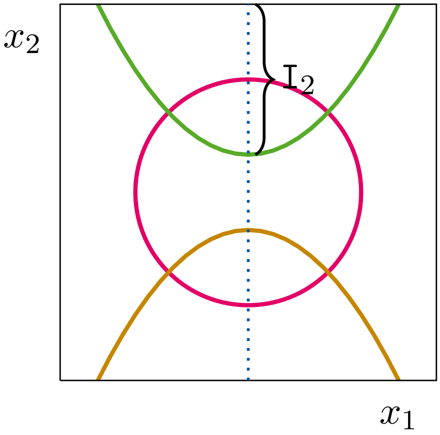

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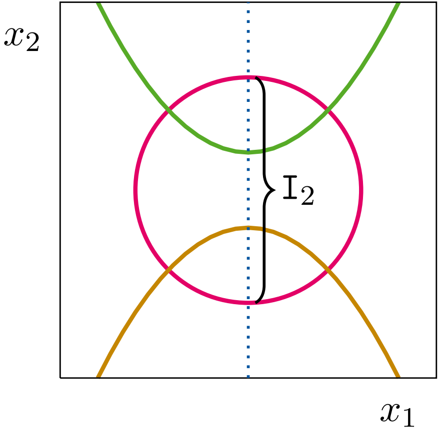

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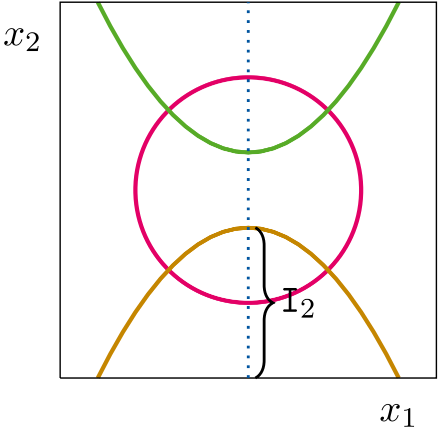

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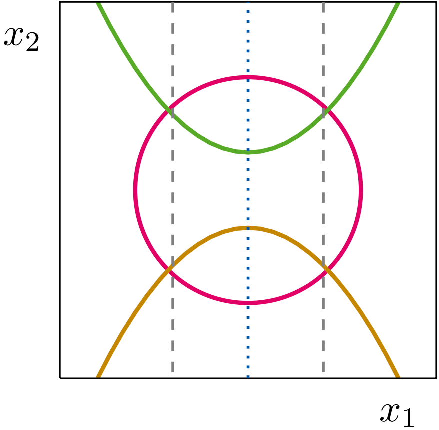

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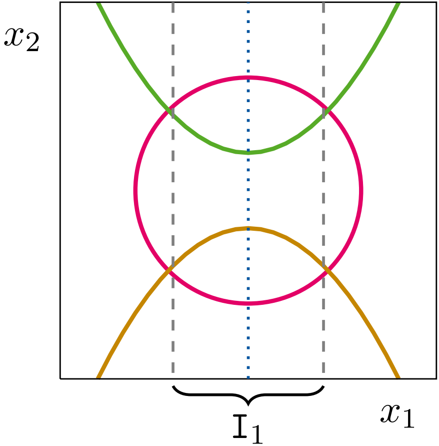
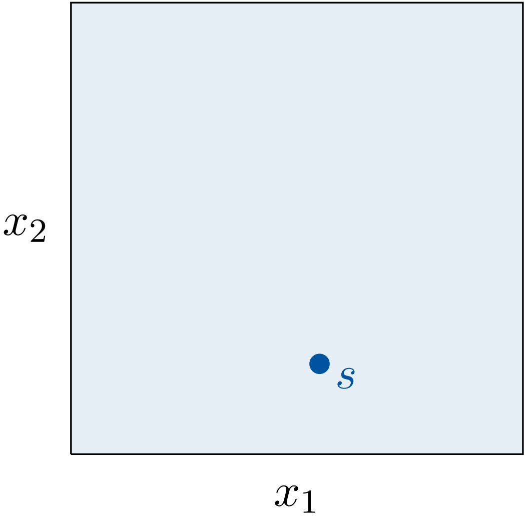


Refinement-based single cell [Brown, Košta 2015]



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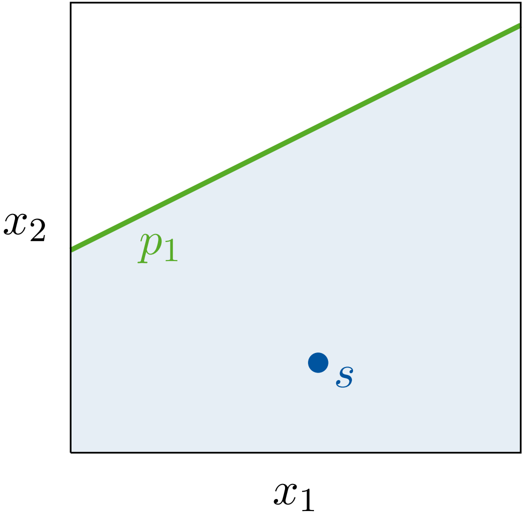

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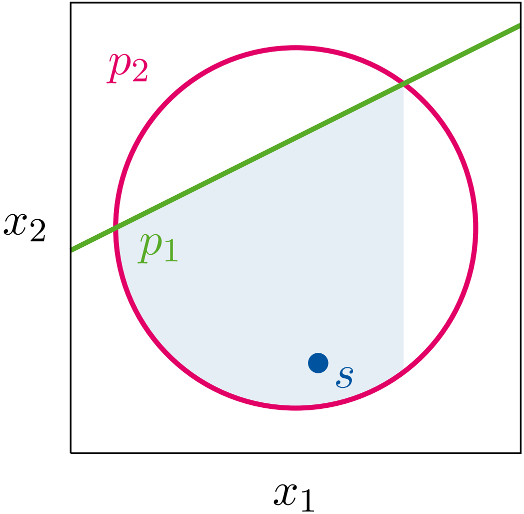

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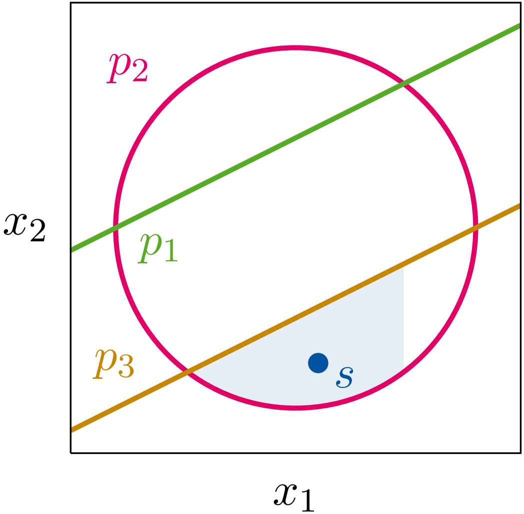

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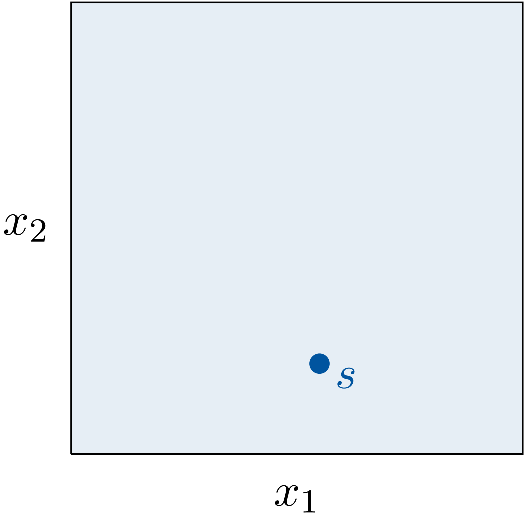

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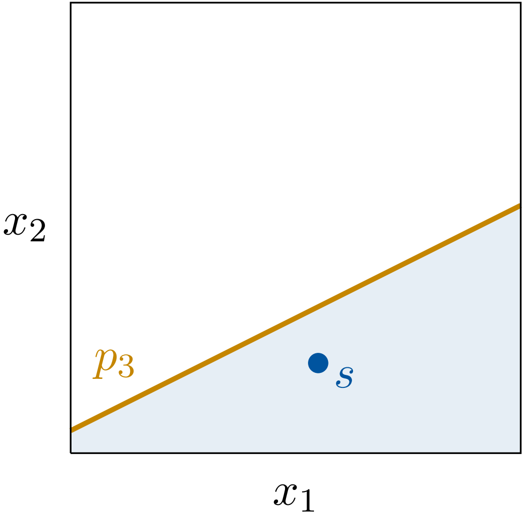

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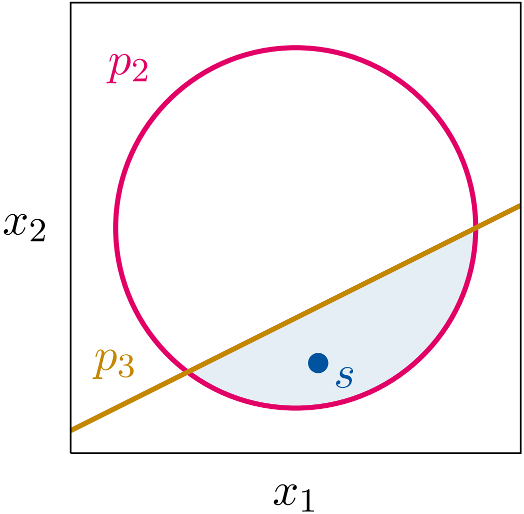

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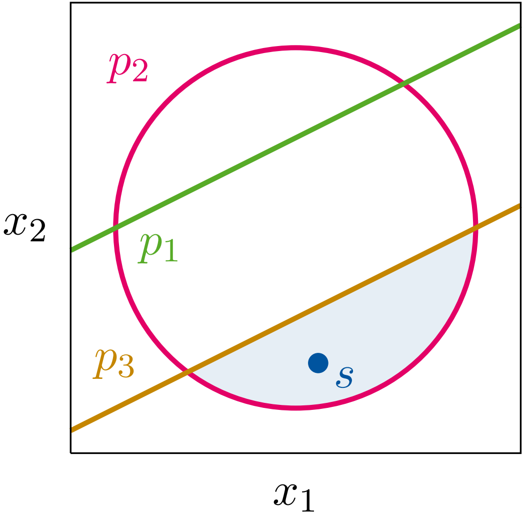

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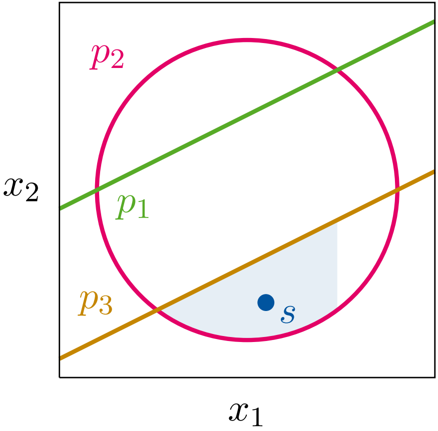

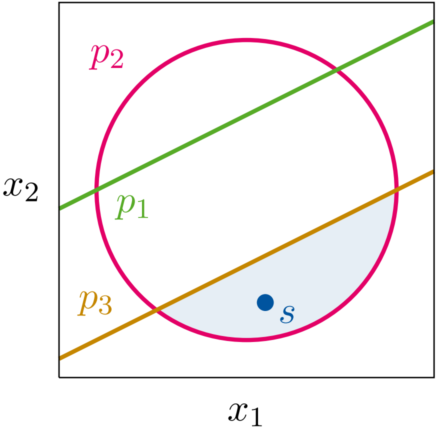

order: p_1, p_2, p_3

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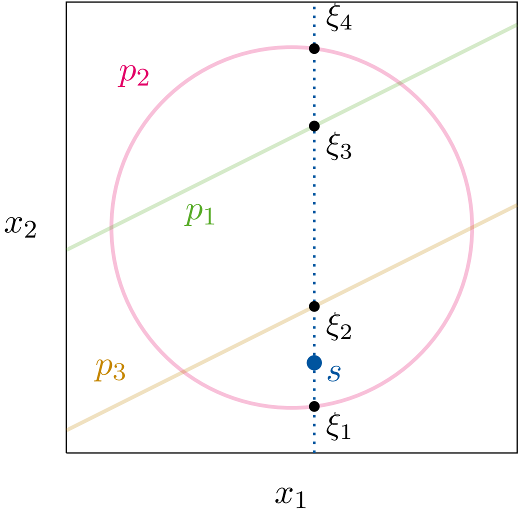
Levelwise single cell construction



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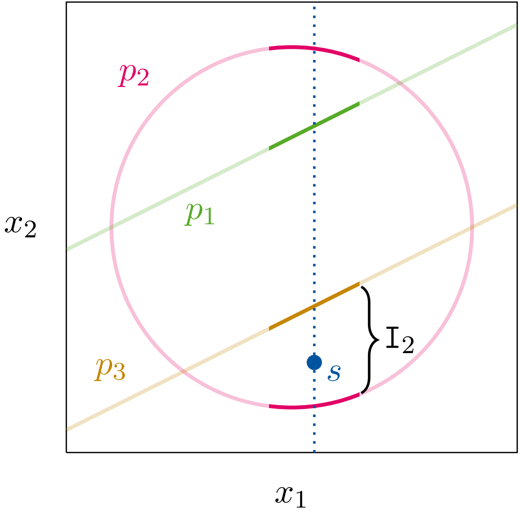

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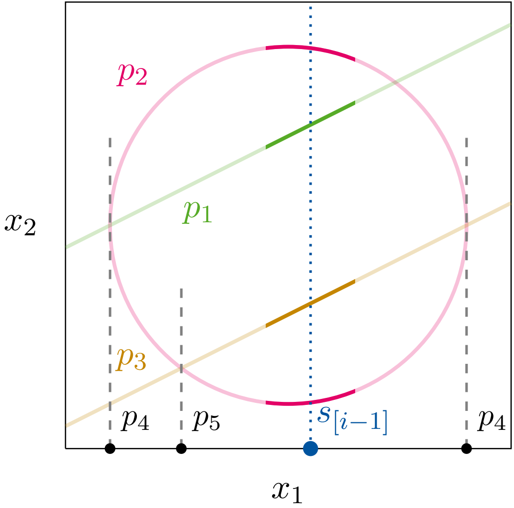

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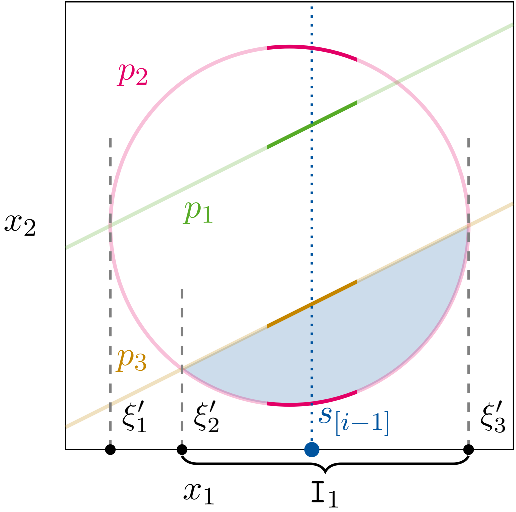

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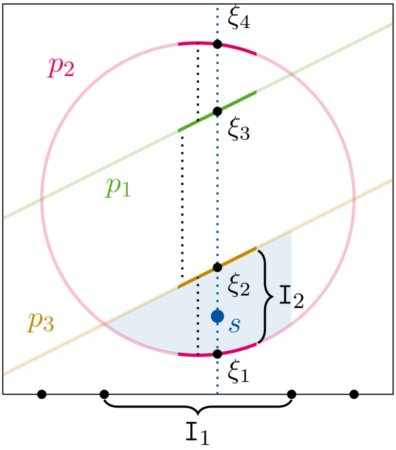
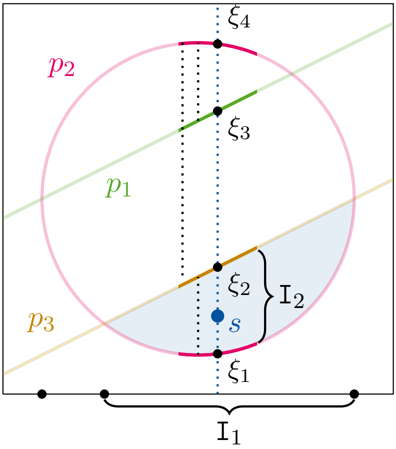


$$\xi_1 \preceq \xi_2 \preceq \xi_3 \preceq \xi_4$$



$$\xi_1 \preceq \xi_2 \begin{smallmatrix} \preceq \xi_3 \\ \preceq \xi_4 \end{smallmatrix}$$

A proof system

$\boxed{\textit{sgn\_inv}(p_1)} \to \textit{sample}(\boxed{s}), \textit{repr}(\boxed{\mathtt{I}_2}, s_1), \textit{ir\_ord}(\boxed{\preceq}, s_1), \textit{an\_proj\_del}(p_1), \textit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_1)), \textit{an\_sub}(1), \textit{connected}(1)$

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$\boxed{\textit{sgn\_inv}(p_3)} \to \textit{sample}(\boxed{s}), \textit{repr}(\boxed{\mathtt{I}_2}, s_1), \textit{ir\_ord}(\boxed{\preceq}, s_1), \textit{an\_proj\_del}(p_3), \textit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_3)), \textit{an\_sub}(1), \textit{connected}(1)$

$\textit{sample}(\boxed{s}) \to \textit{repr}(\boxed{\mathtt{I}_2}, s_1), \textit{sample}(s_1)$

$\textit{repr}(\boxed{\mathtt{I}_2}, s_1) \to \boxed{R = \textit{setOf}(R\downarrow_{[1]}), \boxed{\mathtt{I}_2}}, \textit{an\_proj\_del}(p_2), \textit{an\_proj\_del}(p_3), \textit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_2)), \textit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_3)), \textit{sample}(s_1)$

$\textit{ir\_ord}(\boxed{\preceq}, s_1) \to \textit{an\_proj\_del}(p_1), \textit{an\_proj\_del}(p_2), \textit{an\_proj\_del}(p_3), \textit{ord\_inv}(\mathrm{res}_{x_2}(p_3, p_1)), \textit{ord\_inv}(\mathrm{res}_{x_2}(p_3, p_2)), \textit{an\_sub}(1), \textit{connected}(1), \textit{sample}(s_1)$

$\textit{an\_proj\_del}(p_1) \to \textit{non\_null}(p_1), \textit{ord\_inv}(\mathrm{disc}_{x_2}(p_1)), \textit{an\_sub}(1), \textit{connected}(1)$

$\textit{an\_proj\_del}(p_2) \to \textit{non\_null}(p_2), \textit{ord\_inv}(\mathrm{disc}_{x_2}(p_2)), \textit{an\_sub}(1), \textit{connected}(1)$

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$\textit{non\_null}(p_1) \to$ trivial

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$\textit{ord\_inv}(\mathrm{disc}_{x_2}(p_1)) \to$ trivial

$\textit{ord\_inv}(\mathrm{disc}_{x_2}(p_2)) \to \textit{sgn\_inv}(p_4), \textit{sample}(s_1)$

$\textit{ord\_inv}(\mathrm{disc}_{x_2}(p_3)) \to$ trivial

$\textit{sgn\_inv}(p_4) \to \textit{repr}(\boxed{\mathtt{I}_1})$

$\textit{ord\_inv}(\mathrm{res}_{x_2}(p_3, p_1)) \to$ trivial

$\textit{ord\_inv}(\mathrm{res}_{x_2}(p_3, p_2)) \to \textit{sgn\_inv}(p_5), \textit{sample}(s_1)$

$\textit{sgn\_inv}(p_5) \to \textit{repr}(\boxed{\mathtt{I}_1})$

$\textit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_1)) \to$ trivial

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$\textit{an\_sub}(1) \to \textit{repr}(\boxed{\mathtt{I}_1})$

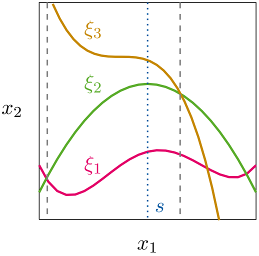
$\textit{connected}(1) \to$ trivial

$\textit{sample}(s_1) \to \textit{repr}(\boxed{\mathtt{I}_1})$

$\textit{repr}(\boxed{\mathtt{I}_1}) \to \boxed{R\downarrow_{[1]} = \textit{setOf}(\boxed{\mathtt{I}_1})}$

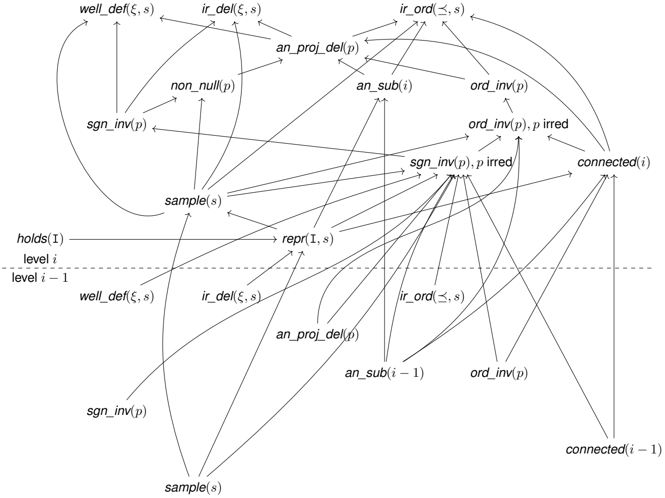
An exemplary proof rule

$i \in \mathbb{N}$, $s \in \mathcal{R}^i$,
$P \subseteq \mathbb{Q}[x_1, \ldots, x_{i+1}] \setminus \mathbb{Q}[x_1, \ldots, x_i]$ irreducible,
no $p \in P$ is nullified on s,
$\Xi \subseteq \mathrm{irExpr}(P, s)$,
$\preceq \subseteq \Xi \times \Xi$ indexed root ordering on Ξ for s



$$\textit{pre}(\textit{ir\_ord}(\preceq, s)) = \{\textit{sample}(s), \textit{an\_sub}(i), \textit{connected}(i)\} \cup \{\textit{an\_proj\_del}(\xi.p) \mid \xi \in \Xi\} \cup \{\textit{ord\_inv}(\mathrm{res}_{x_{i+1}}(\xi.p, \xi'.p)) \mid \xi \preceq \xi'\}.$$

A graph of properties



Heuristics

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- Choice of representation `I`

Heuristics

- Application of proof rules
- Choice of representation $\mathtt{I}$
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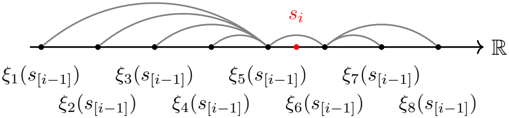
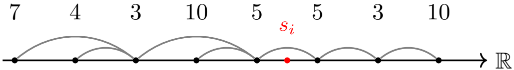
Heuristics: Indexed root orderings

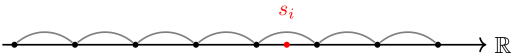
BIGGEST CELL



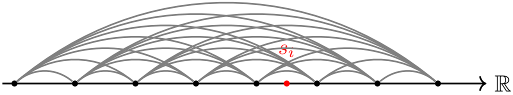
LOWEST DEGREE BARRIERS



CHAIN

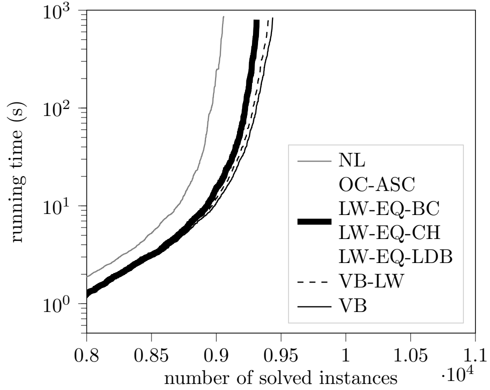


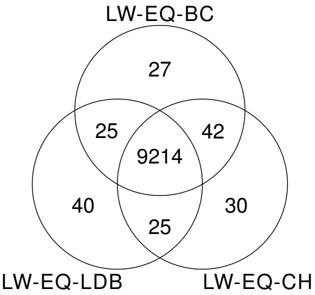
FULL



Experimental results

Implementation in `SMT-RAT-MCSAT`





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Future work

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- Proof generation

Future work

- Heuristic choices: solve combinatorial problems exactly
- Under-approximating cells: insert additional polynomials simplifying cell descriptions
  - Linear descriptions allow integration with incremental linearization
- Compound cylindrical cells: allow multiple lower and upper bounds
- Extend proof system for coverings (and other applications)
- Completeness through Lazard's projection
- Refine rules: mathematical proofs needed
- Proof generation
  - Either in the NLSAT or the CAIC setting