Subtropical Satisfiability for SMT Solving

NASA Formal Methods Symposium (NFM 2023)

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Satisfiability-modulo-theories (SMT)

Boolean satisfiability (SAT) ...

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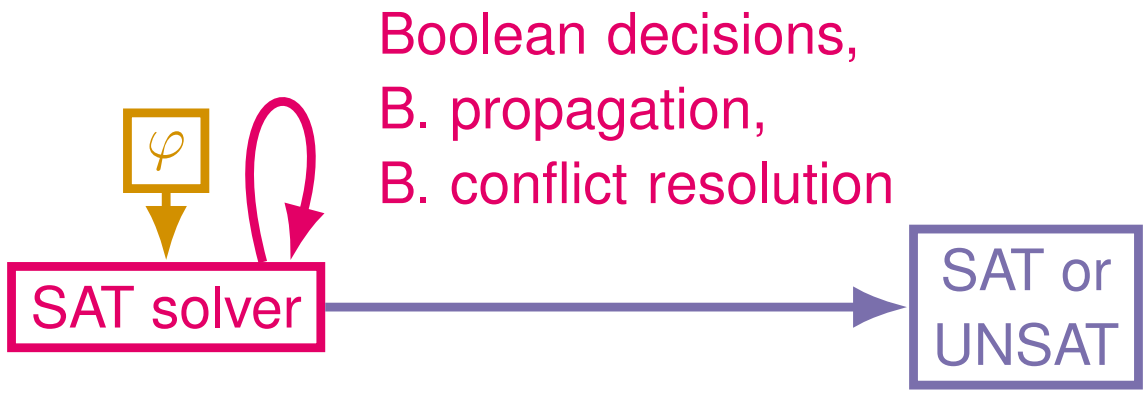

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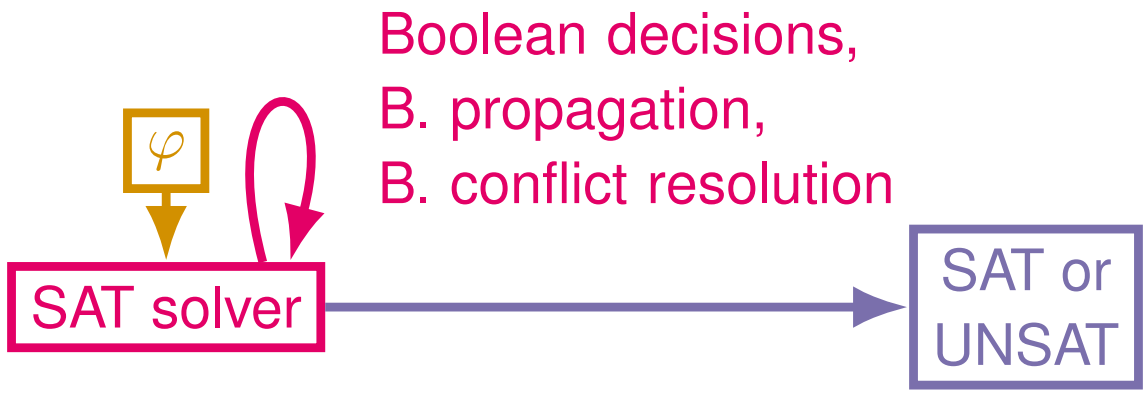

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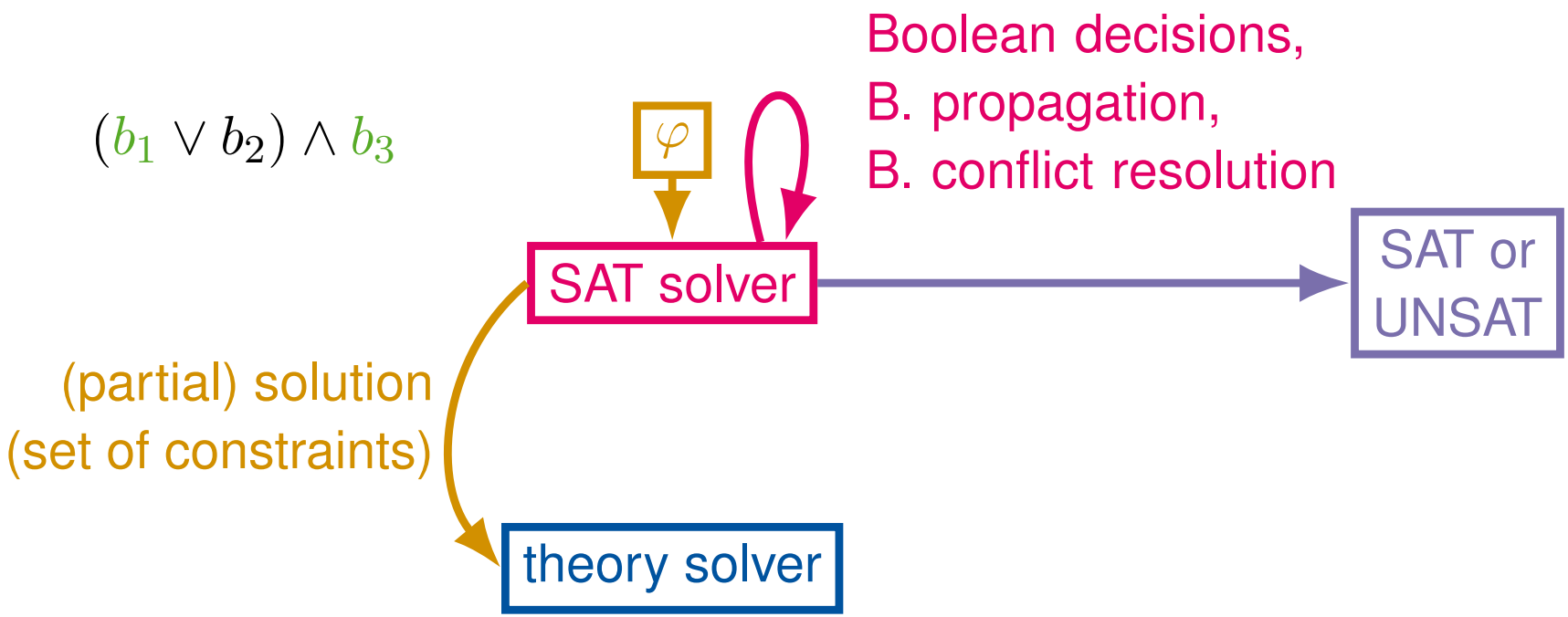

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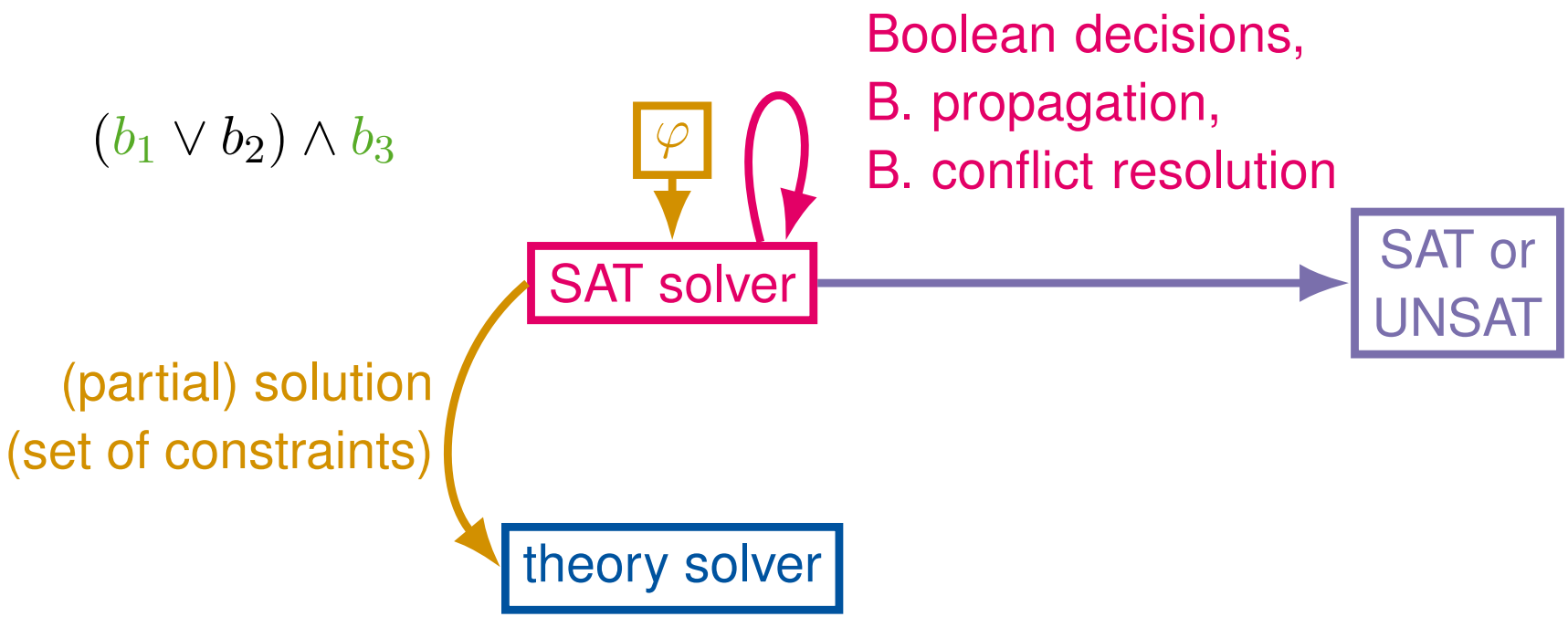

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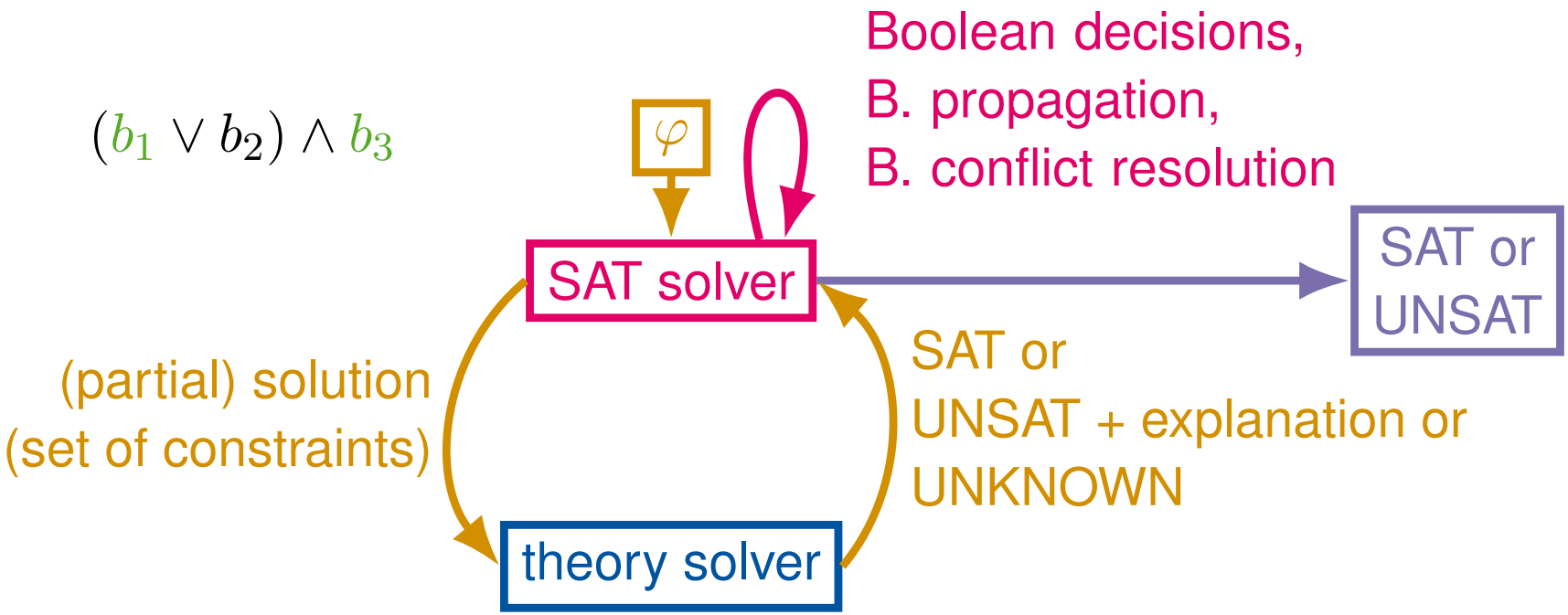

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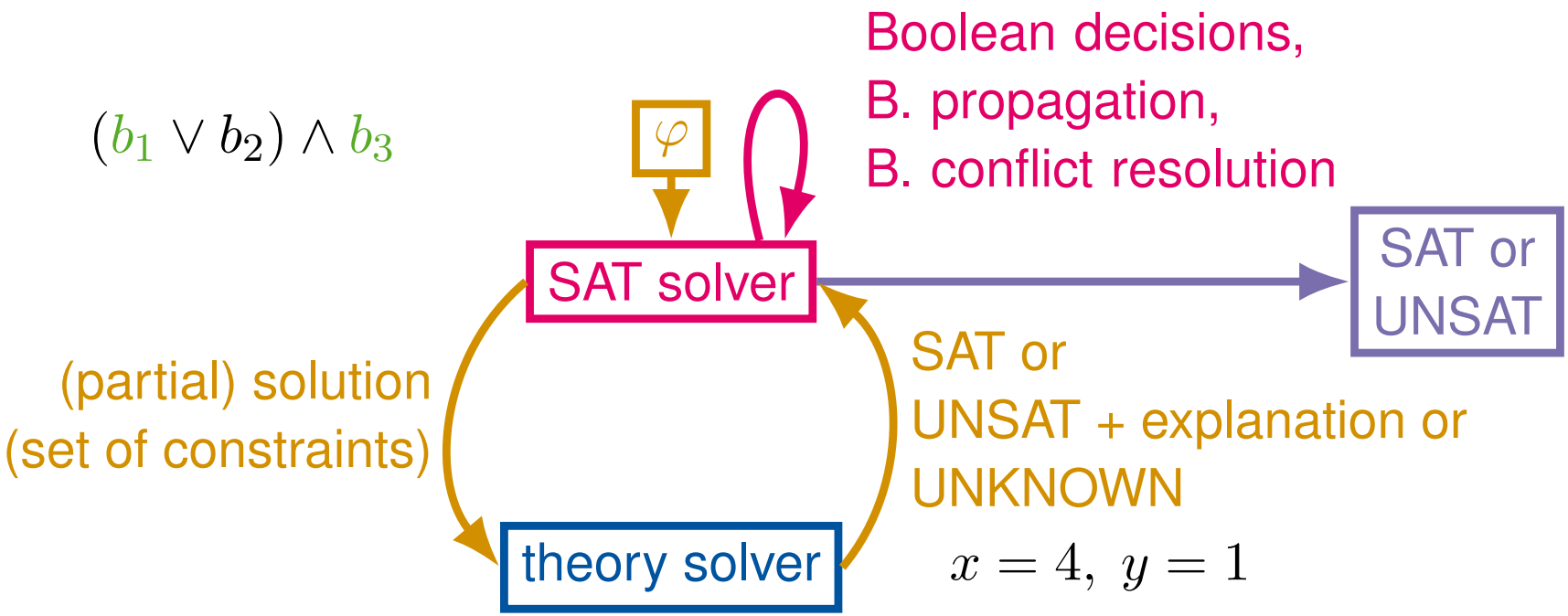

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- Linear real arithmetic is efficiently solved using simplex in practise

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- which finds solutions for sets of inequations or a single equation,
- by encoding them in QFLRA,
- is simple and fast.

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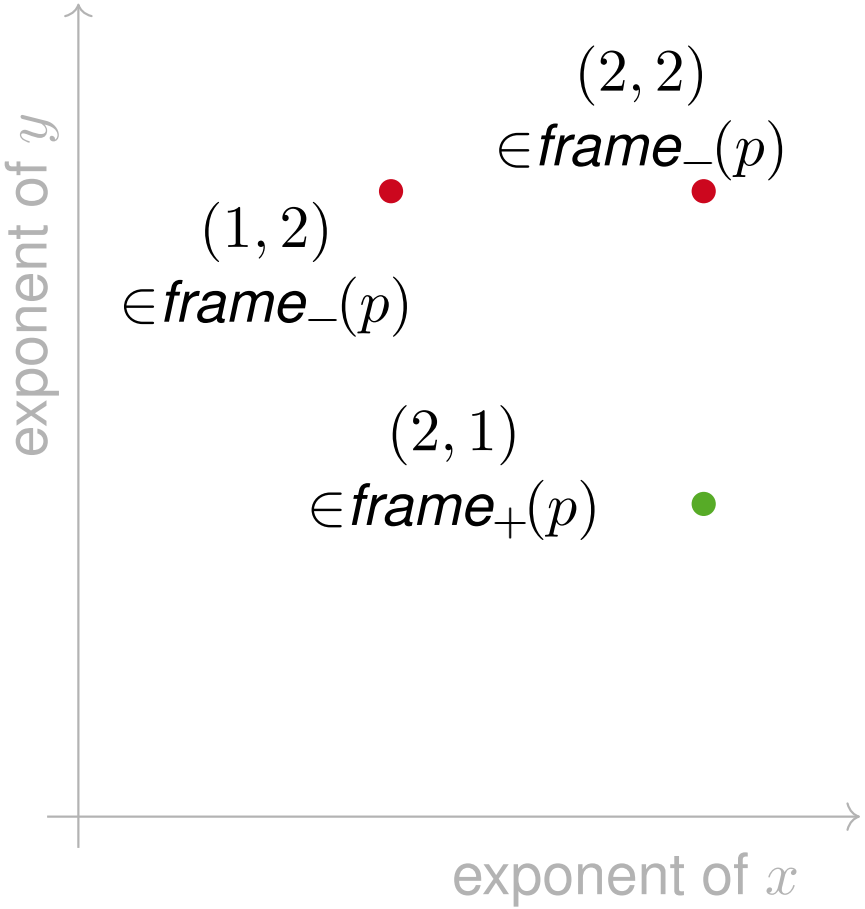

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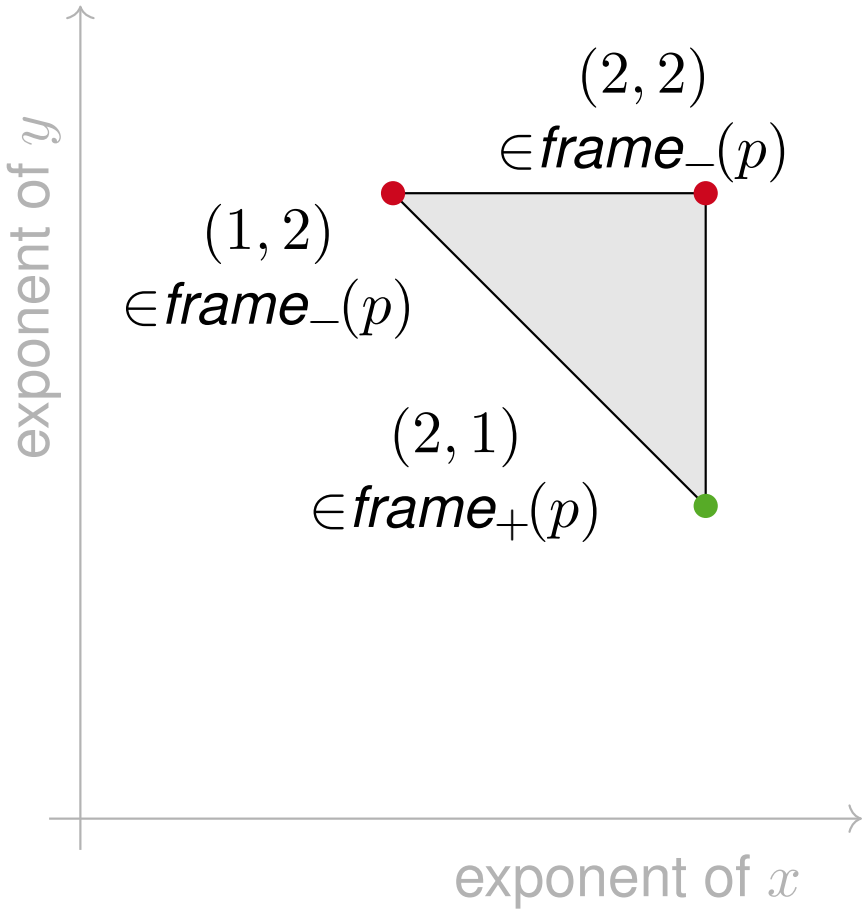

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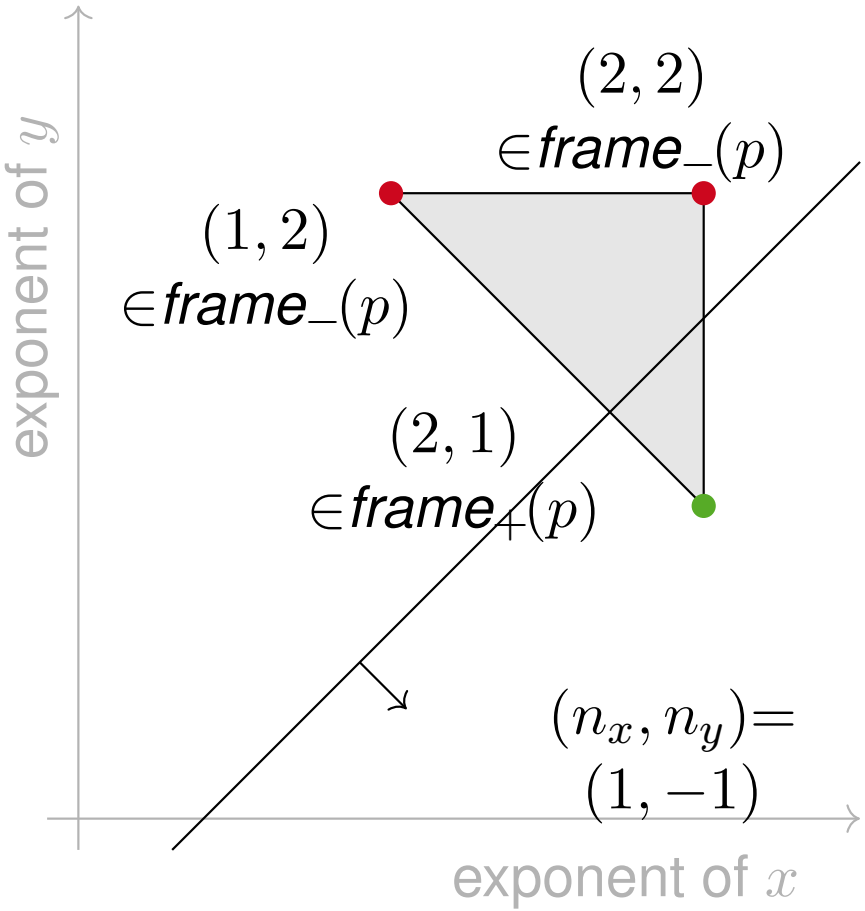

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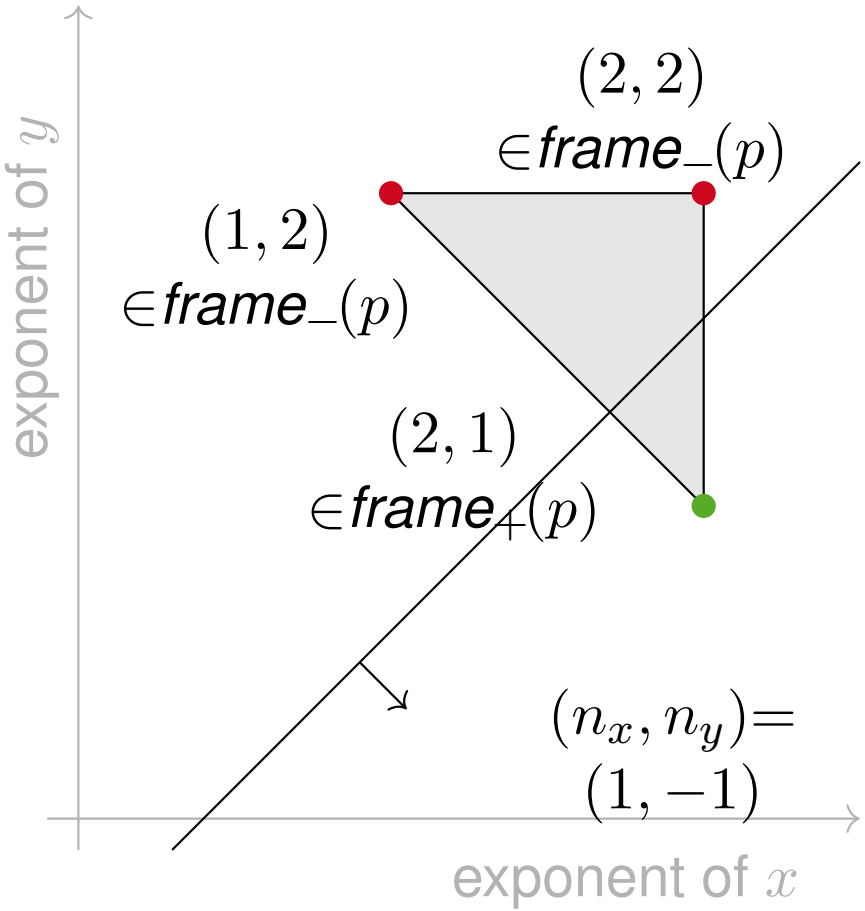

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- $\mathcal{ST}_{p<0}, \mathcal{ST}_{p\geq 0}, \mathcal{ST}_{p\leq 0}, \mathcal{ST}_{p\neq 0}$ can be derived easily

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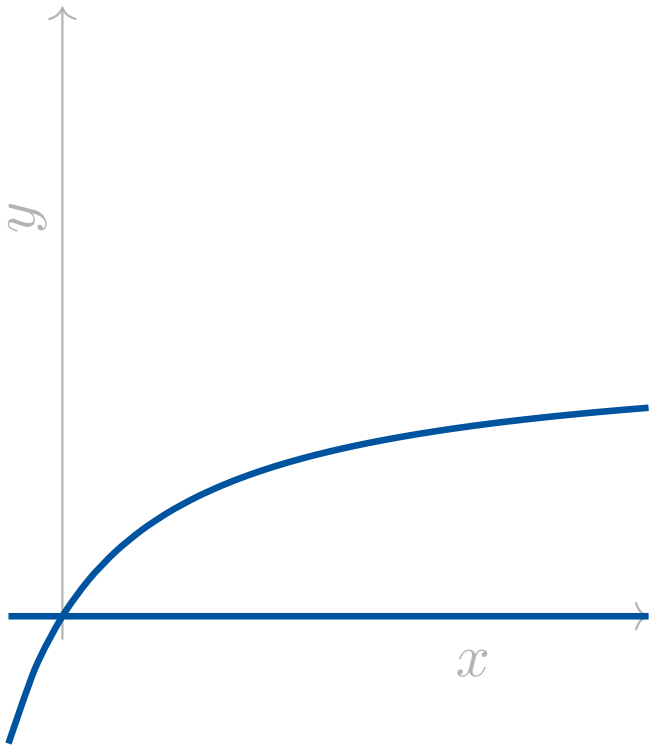

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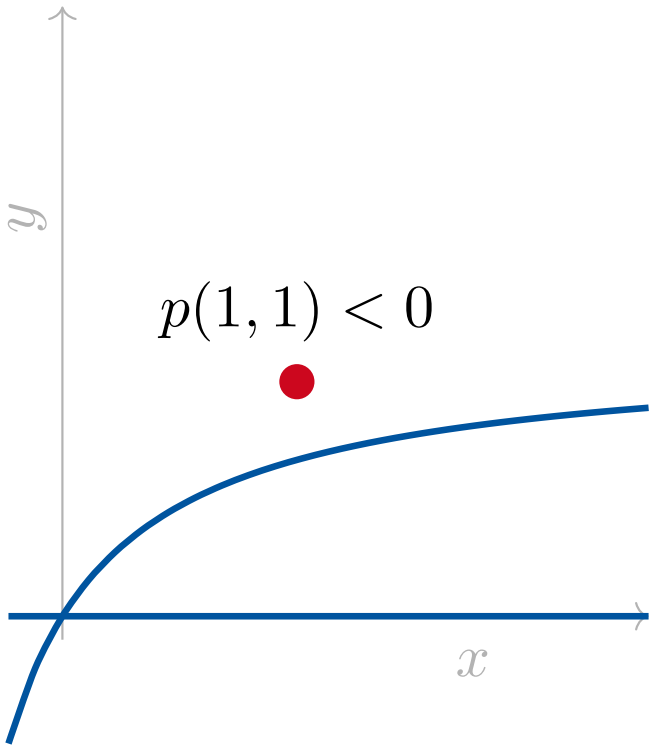

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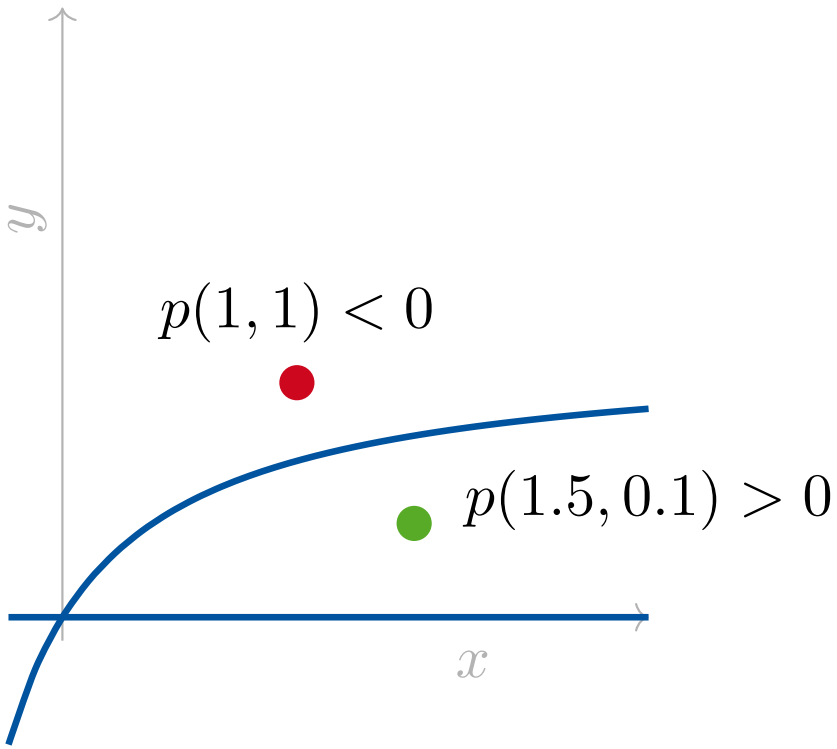

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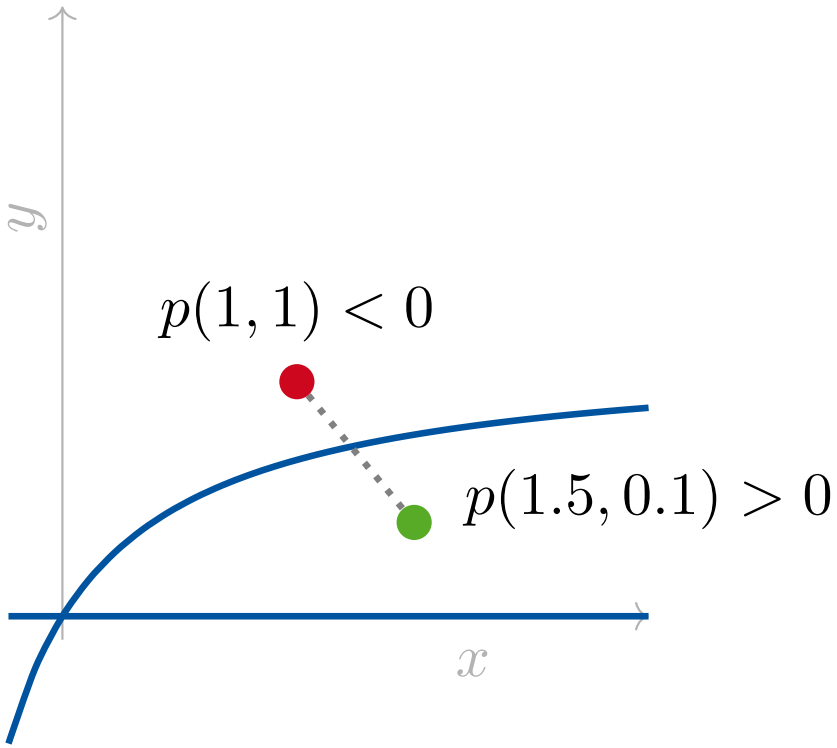

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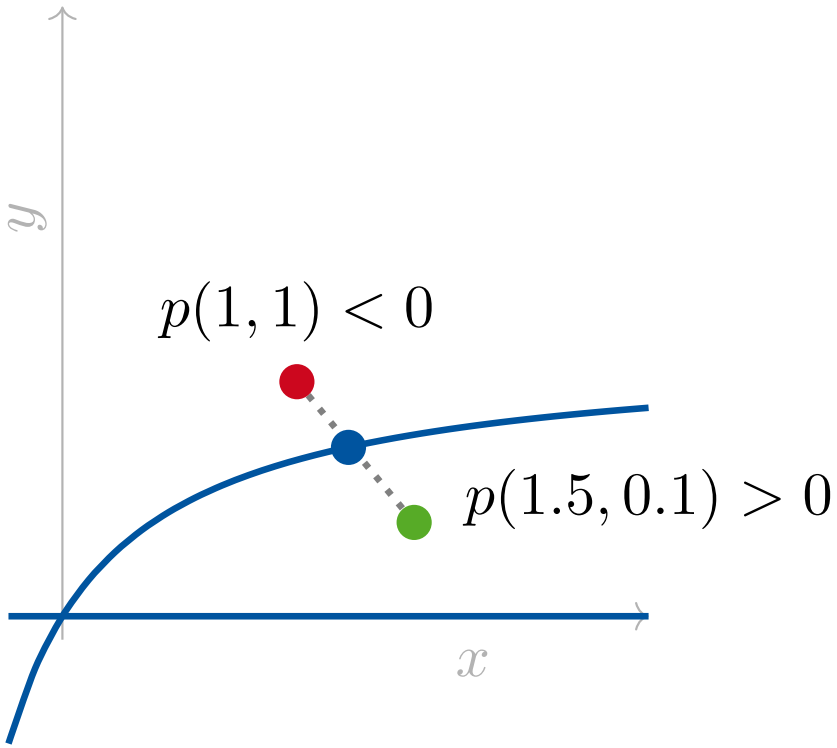

Generalization to formulas

How to make use of subtropical for general QFNRA formulas?

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Transformation (excerpt):

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Subtropical is not applicable to sum-of-square polynomials!

Solutions for conjunctions of inequations [Fontaine et al 2017]

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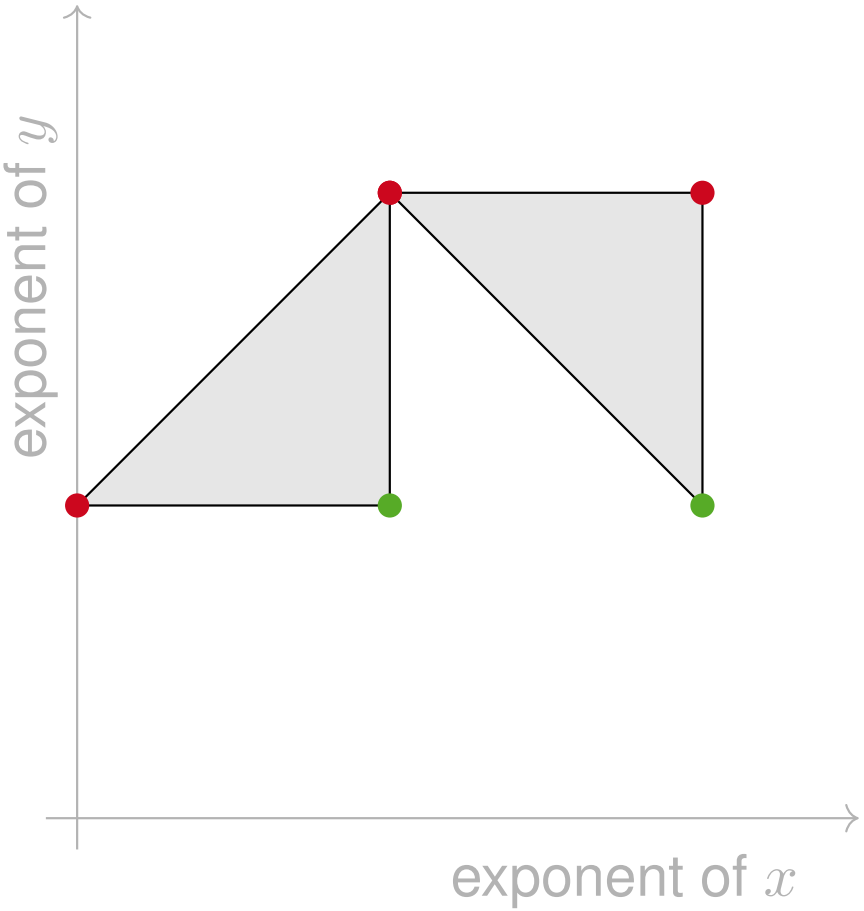

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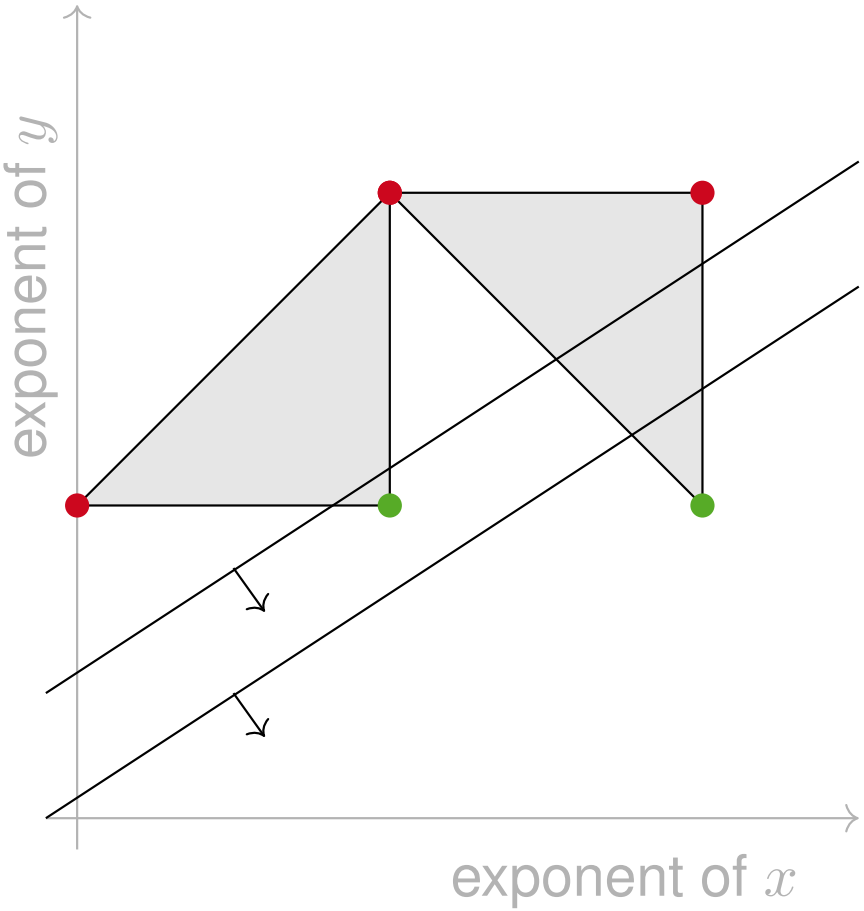

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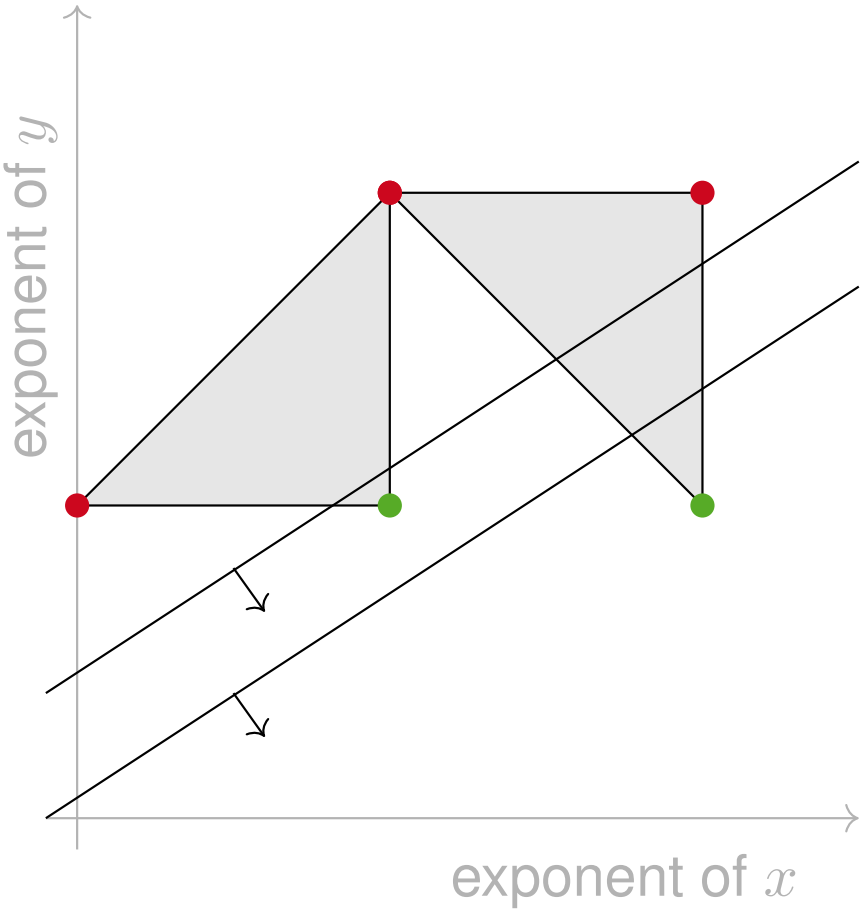

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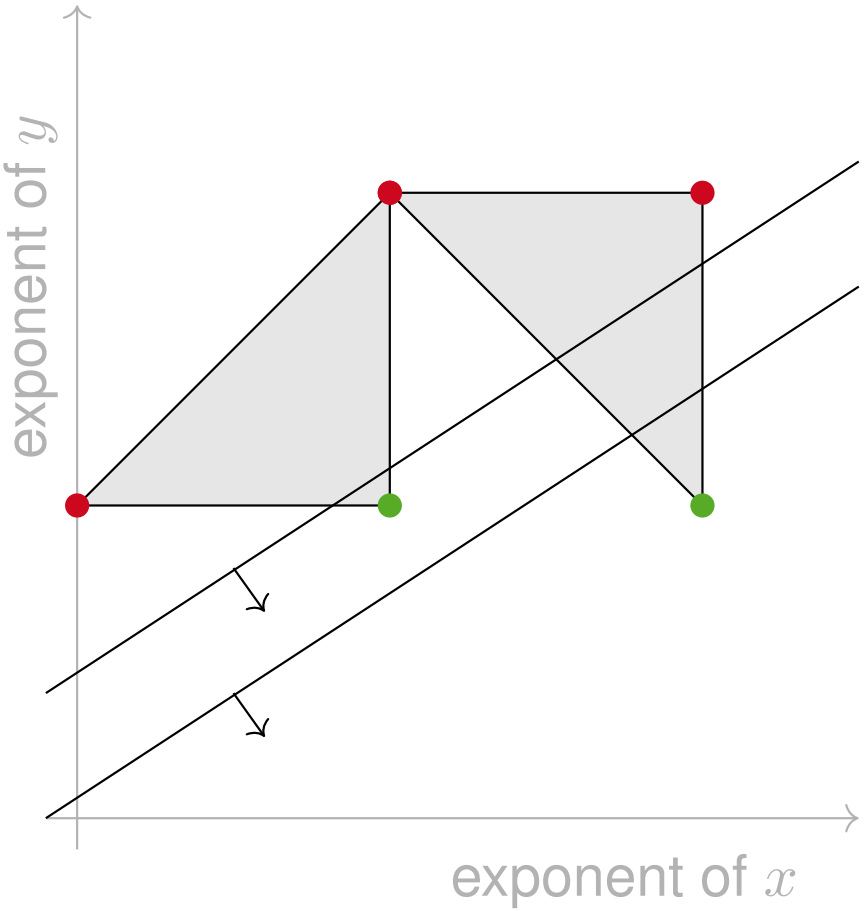

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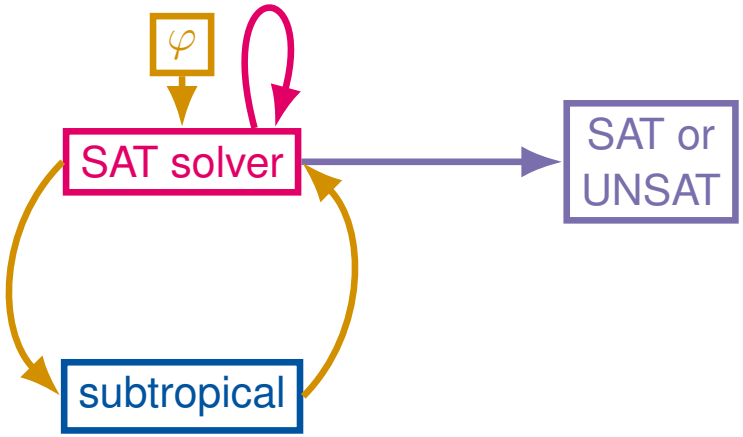

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Alternative: Use auxiliary Boolean variables.

$$\mathcal{ST}^{aux}_{\varphi}(n) := (a_1 \vee a_2) \wedge (a_1 \to \mathcal{ST}_{p_1>0}(n)) \wedge (a_2 \to \mathcal{ST}_{p_2>0}(n))$$

Experiments

- implementation in SMT-RAT
- QF_NRA benchmark set from SMT-LIB, 12134 instances
- Intel® Xeon® Platinum 8160 2.1GHz
- 2 minutes time limit, 4GB memory limit per instance

Results for pure subtropical solvers

- focus on 3580 non-trivial instances (1735 sat, 499 unsat, 1346 unknown)

	Tr (equation)	$\mathcal{ST}$ enc.	$\mathcal{ST}^{aux}$ enc.	DPLL(T)
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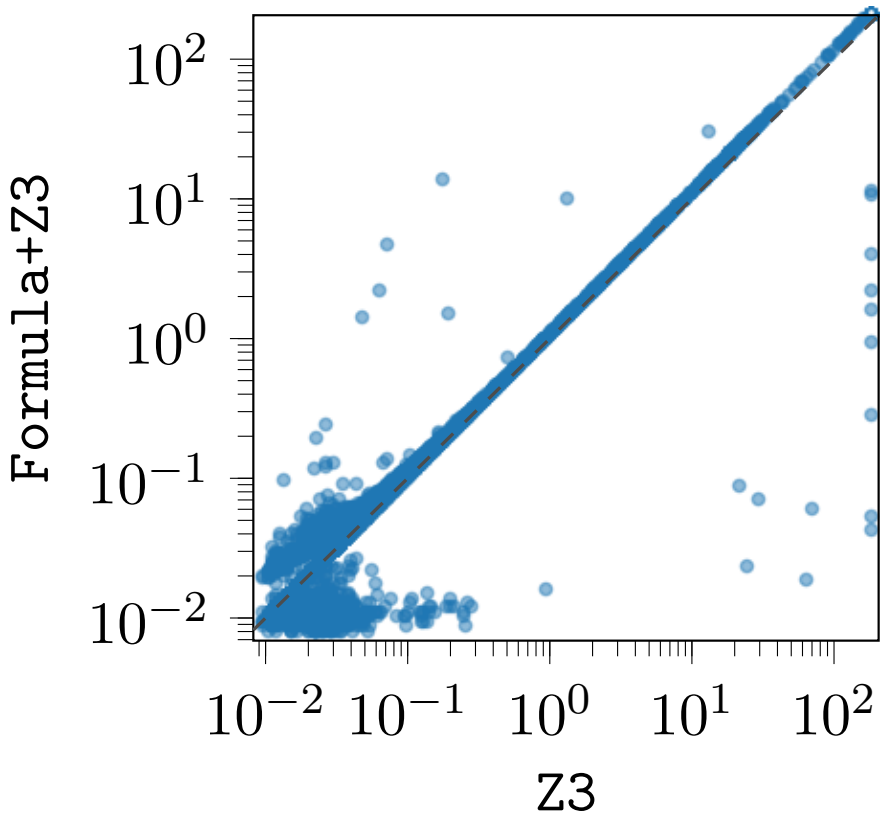
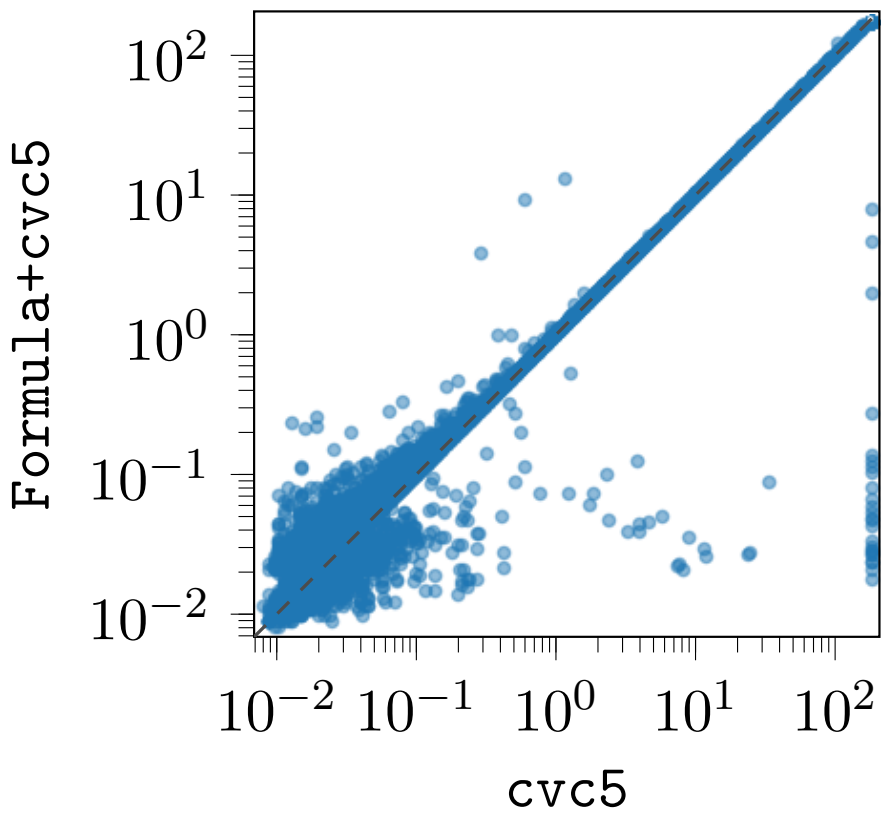
	CAlC	DPLL(T)+CAlC
sat	5055	5071

Combination with z3 and cvc5

- $10s$ for z3/cvc5 on subtropical encoding + $110s$ for z3/cvc5 on original input



9 new benchmarks solved



25 new benchmarks solved

Conclusion

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- ... in particular contains few high-degree polynomials.