

# Merging Adjacent Cells during Single Cell Construction

Jasper Nalbach, Erika Ábrahám

RWTH Aachen University, Germany

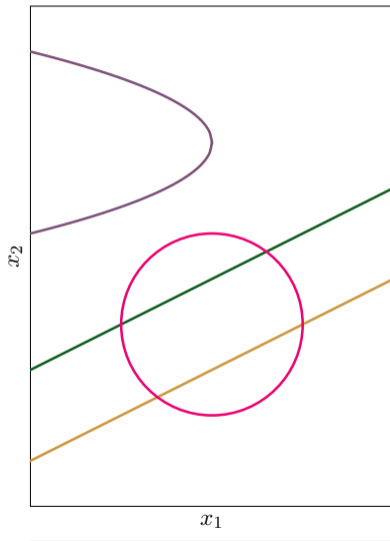
CASC 2024

September 5, 2024



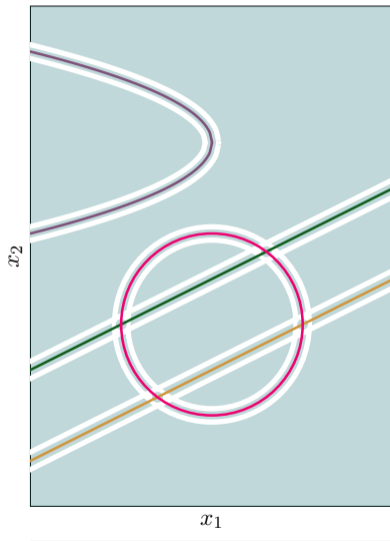
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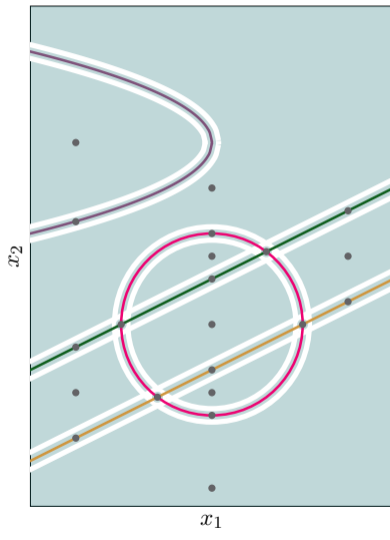
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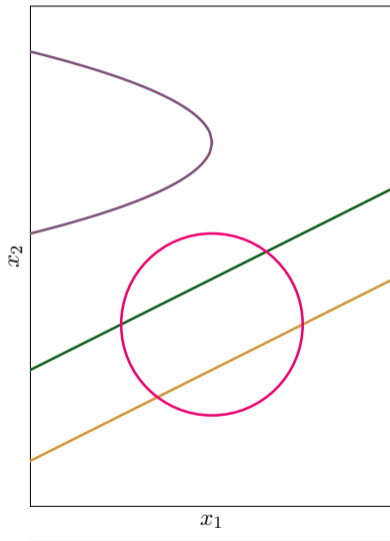
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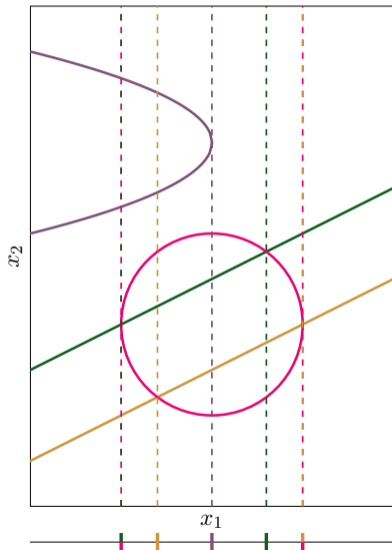
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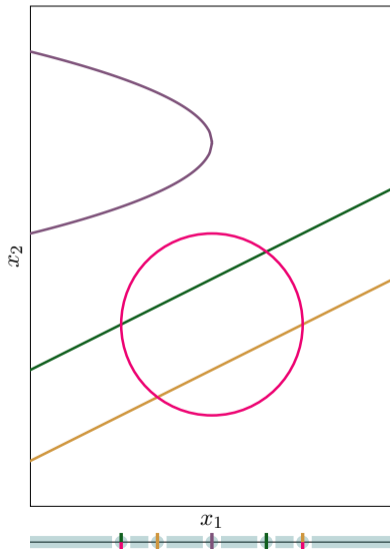
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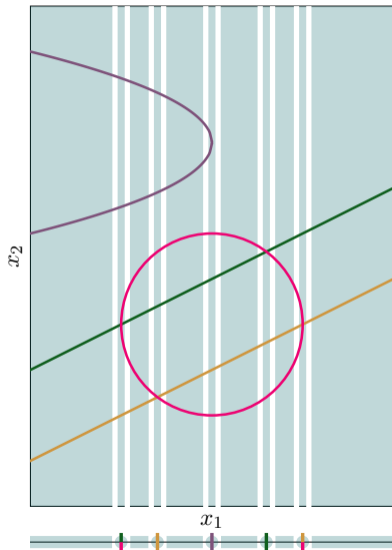
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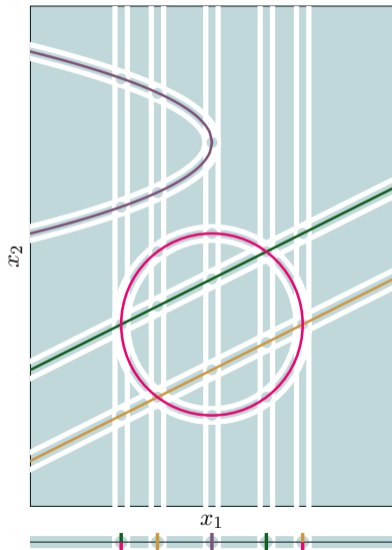
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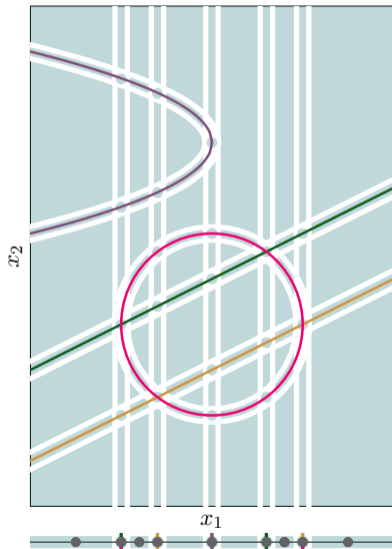
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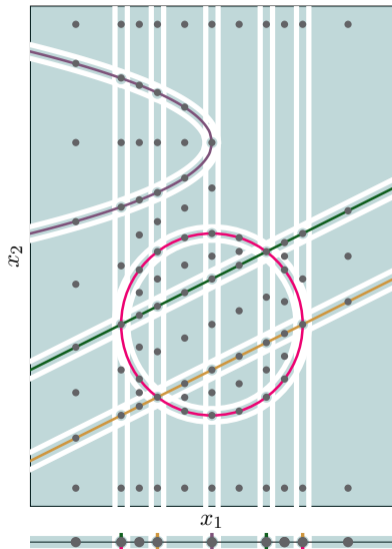
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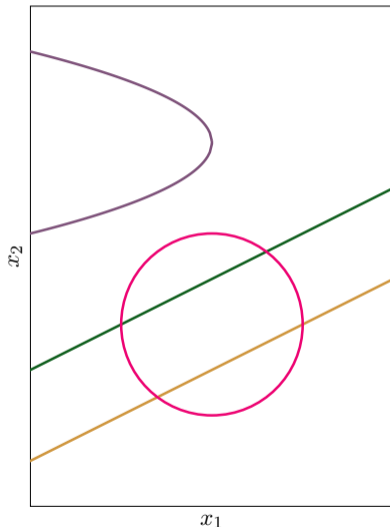
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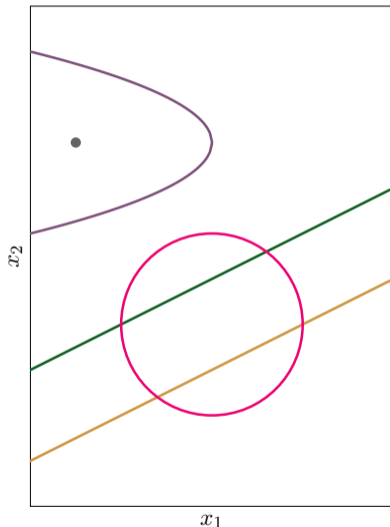
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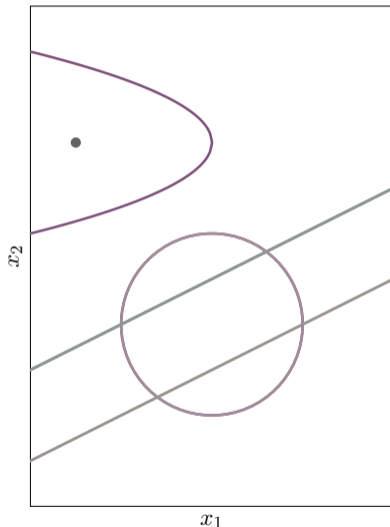
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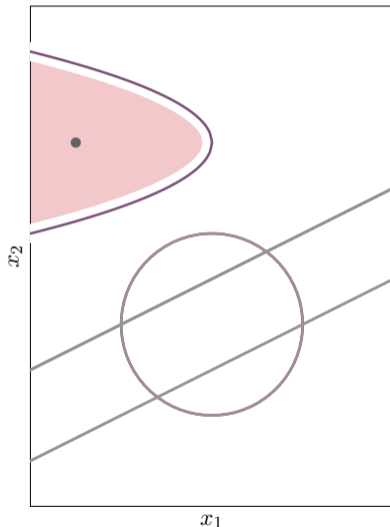
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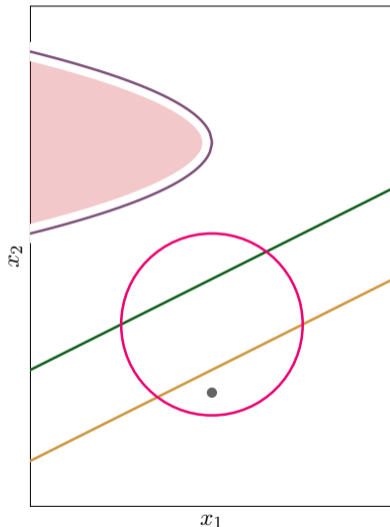
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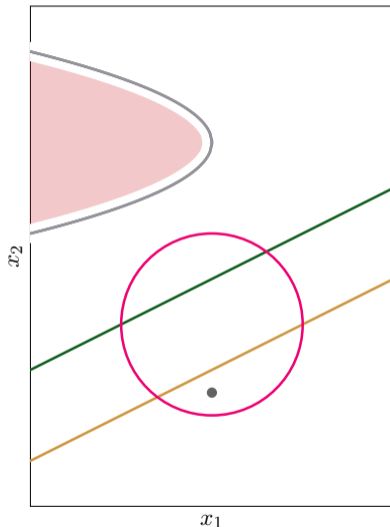




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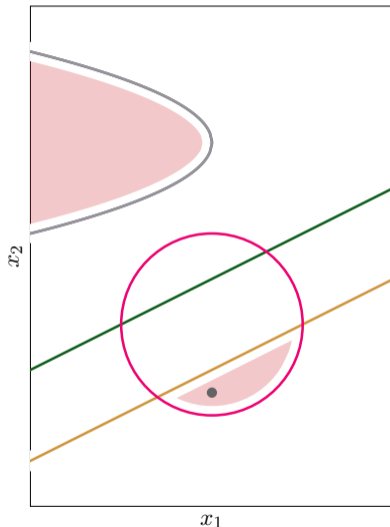
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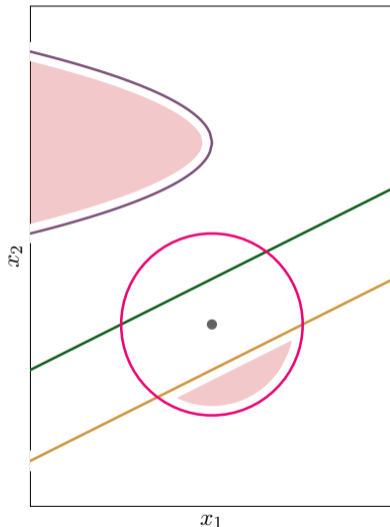
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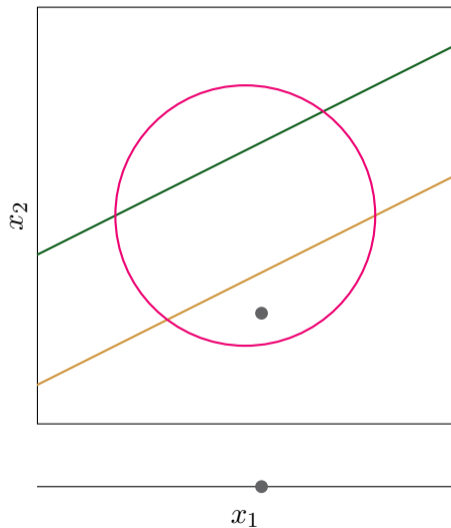
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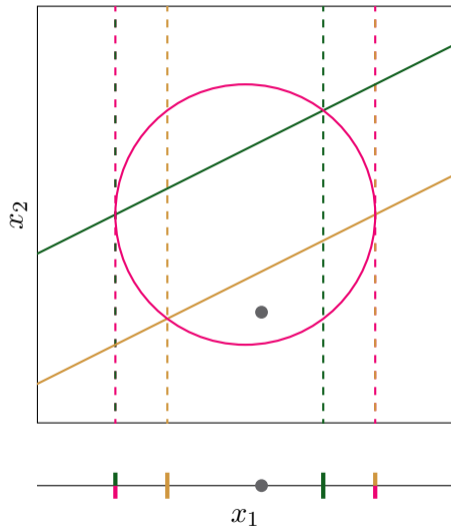
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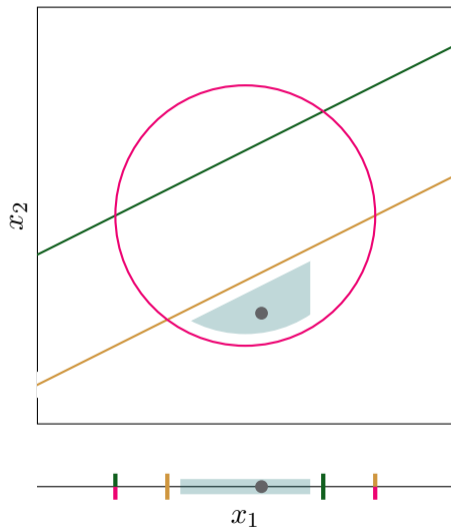
# Naive Single Cell Construction



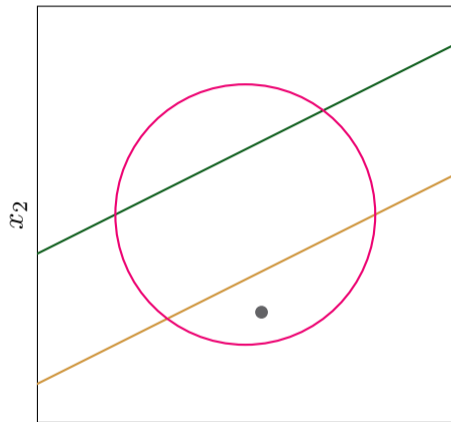
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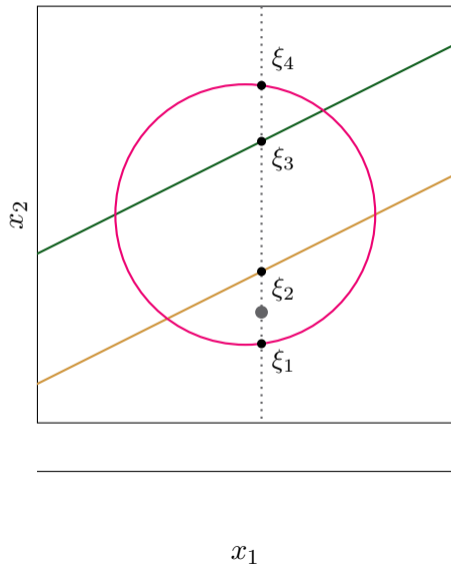
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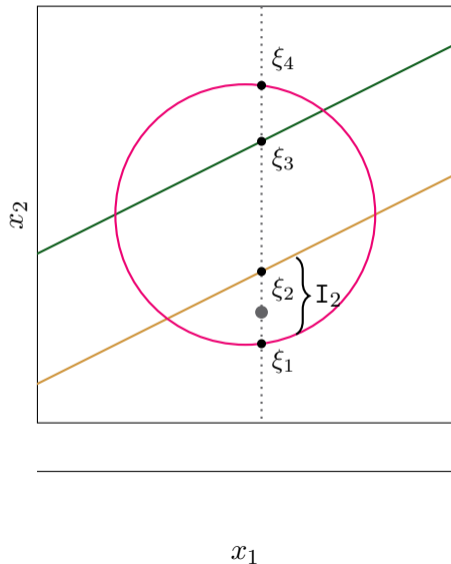


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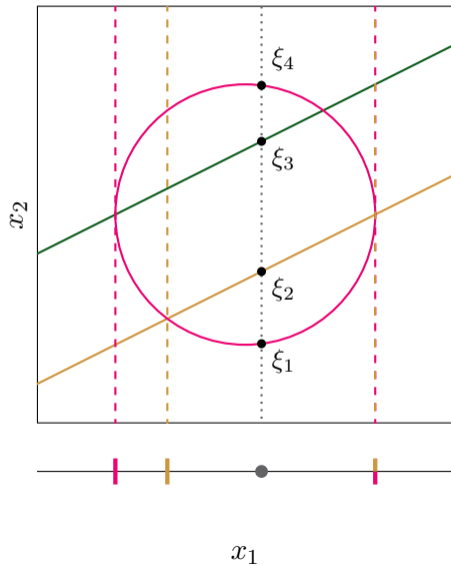




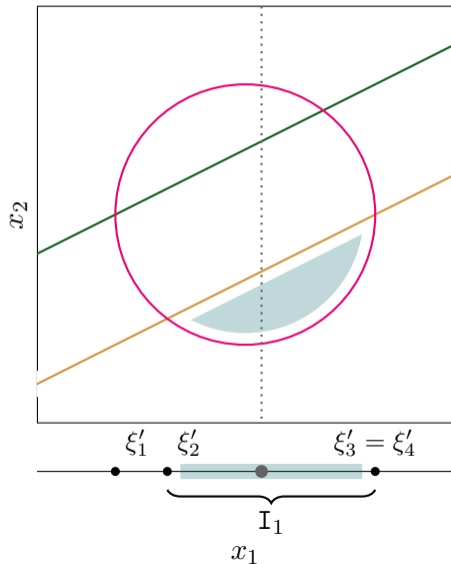
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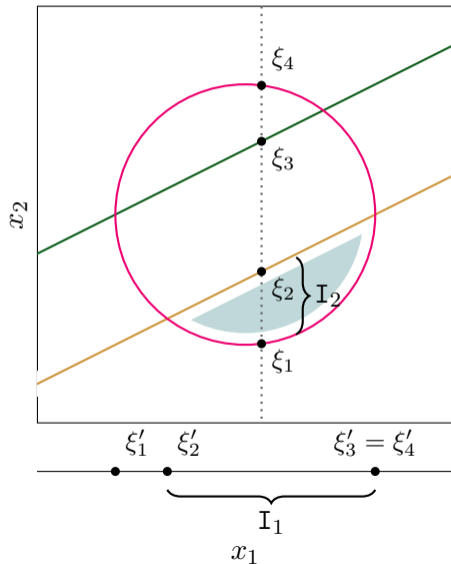
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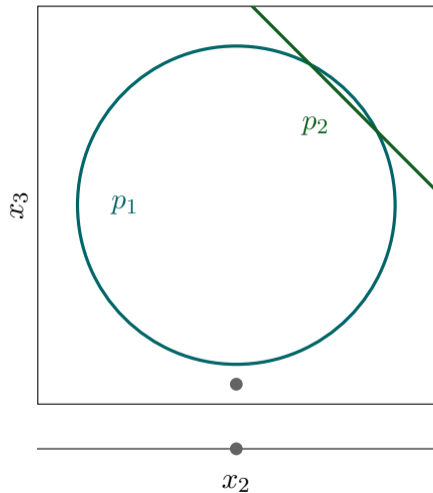
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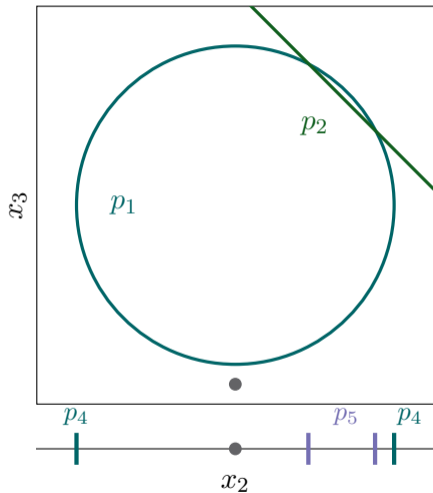
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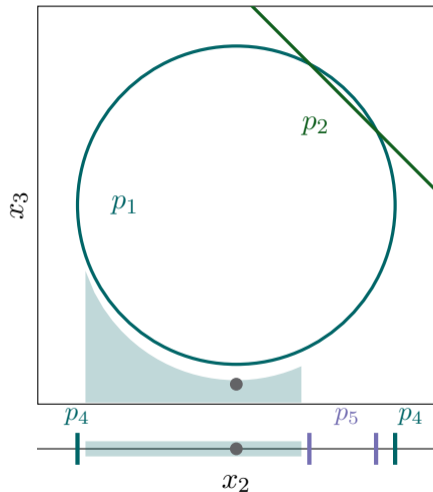
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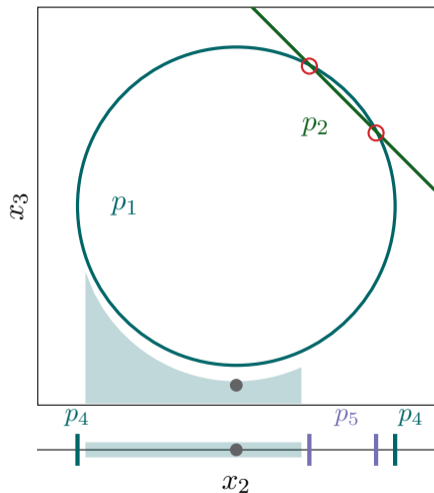
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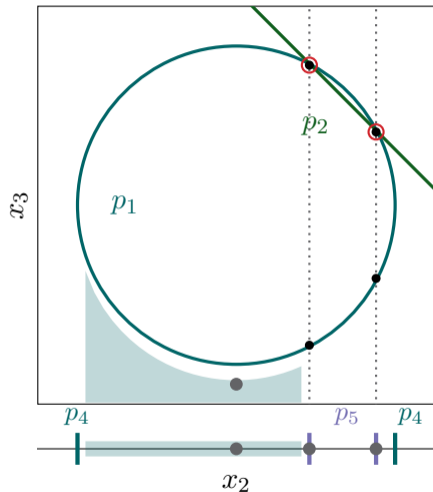


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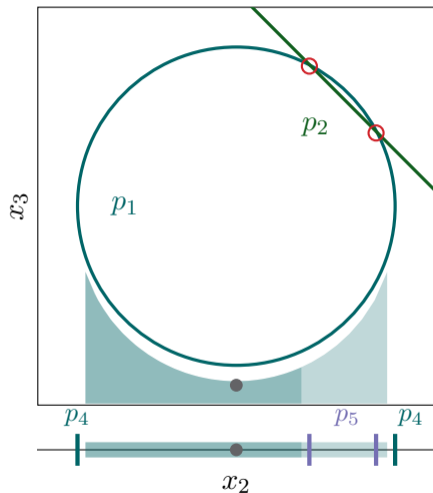




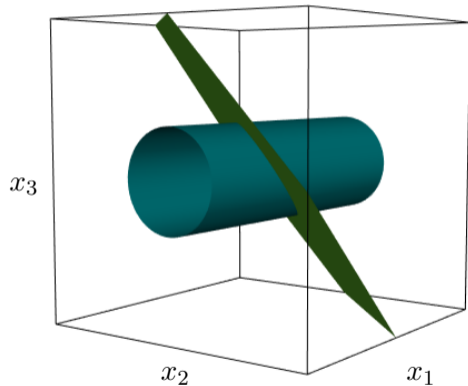
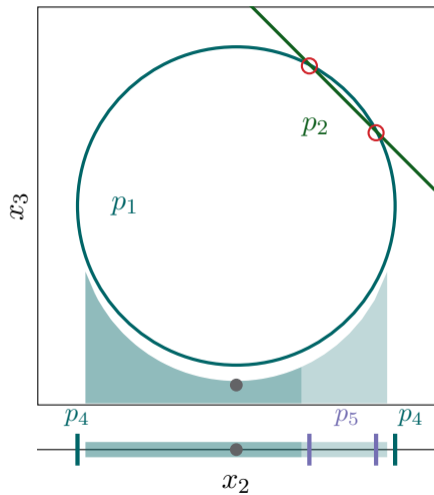
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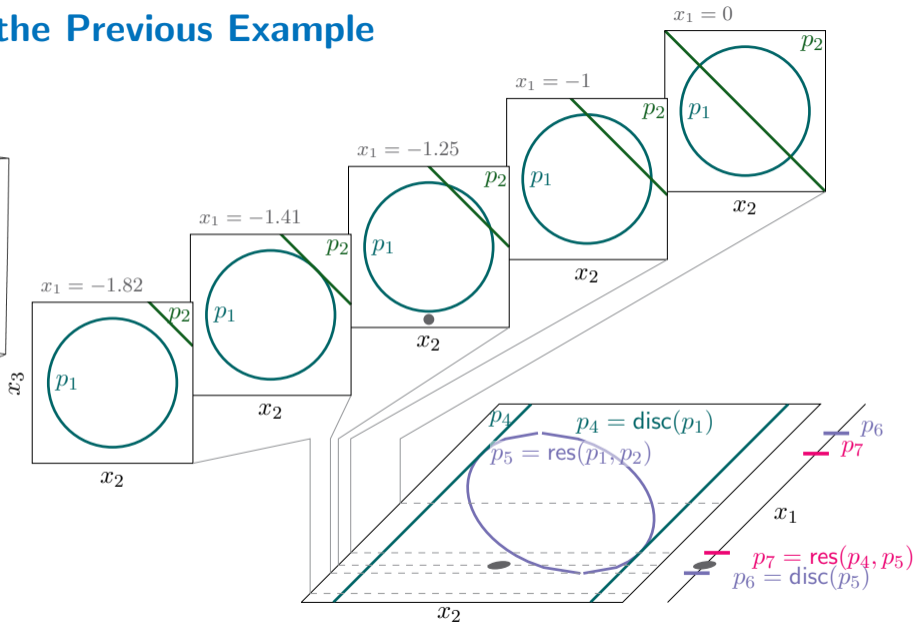
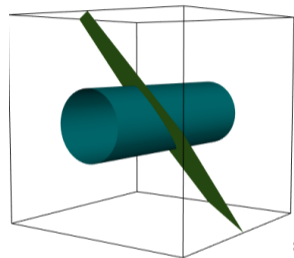
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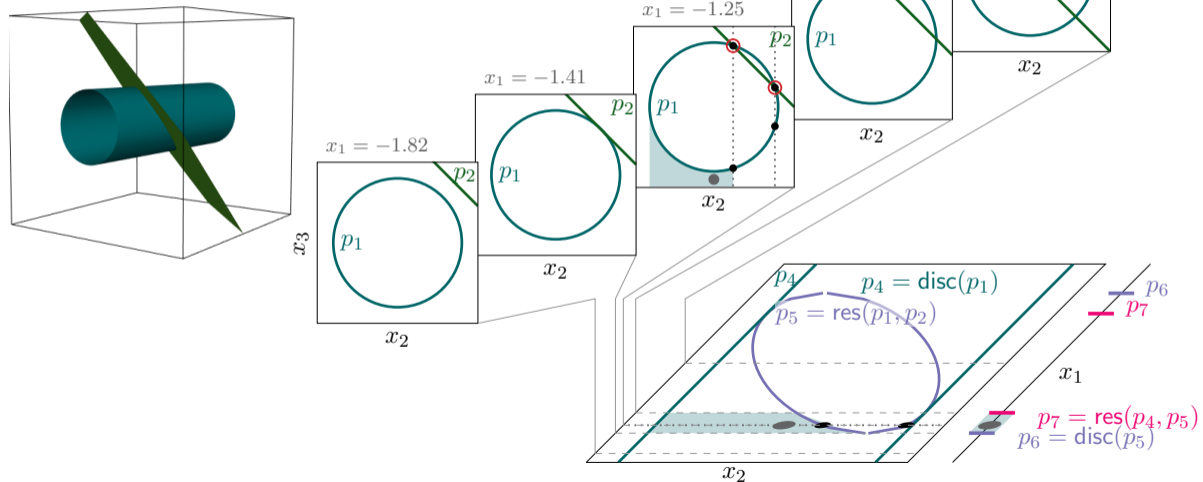
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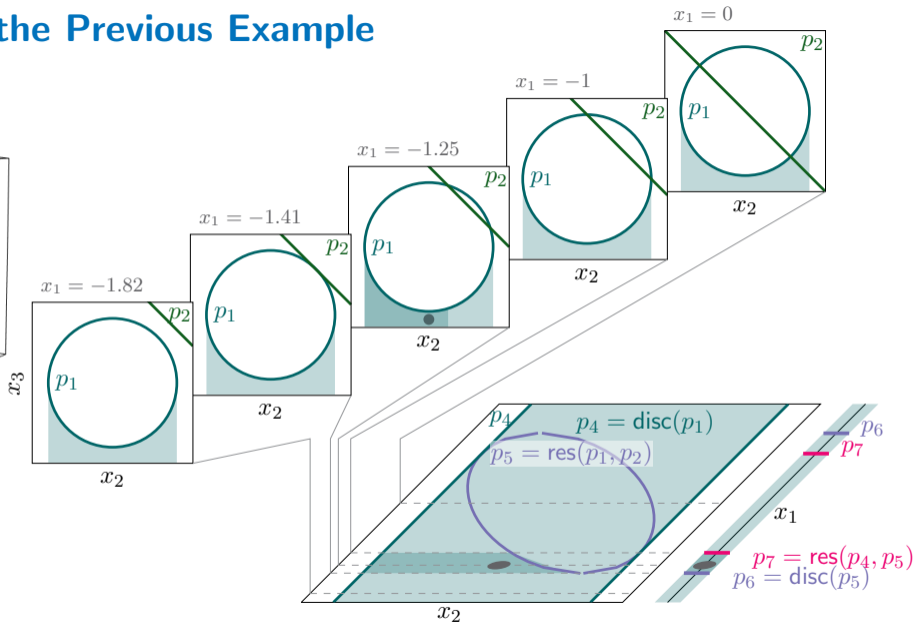
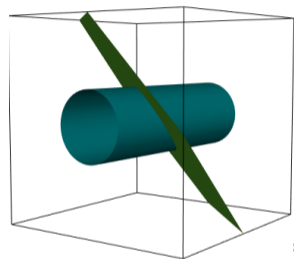
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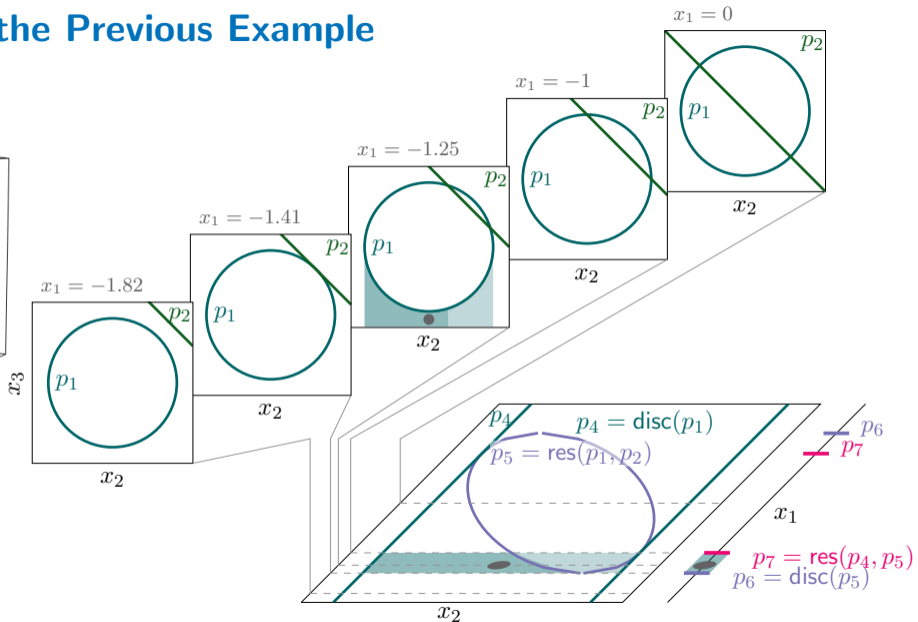
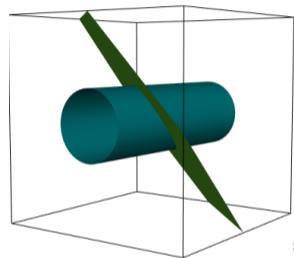
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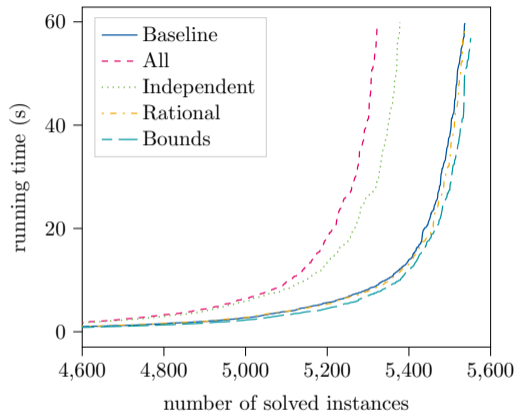
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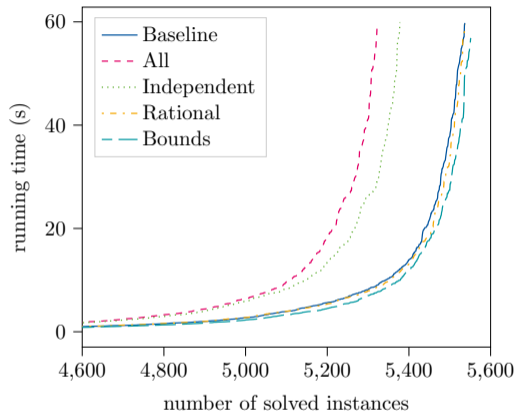
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# Results



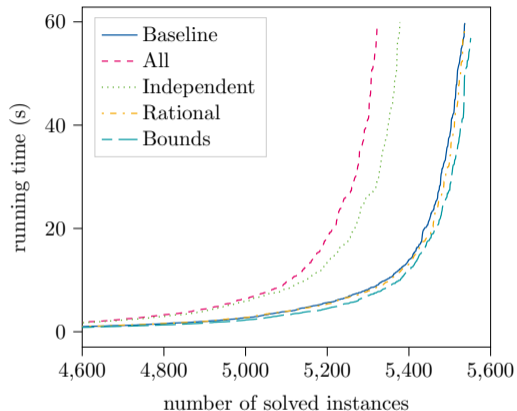


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- ▶ Quality of cells slightly improve when merging is applied.

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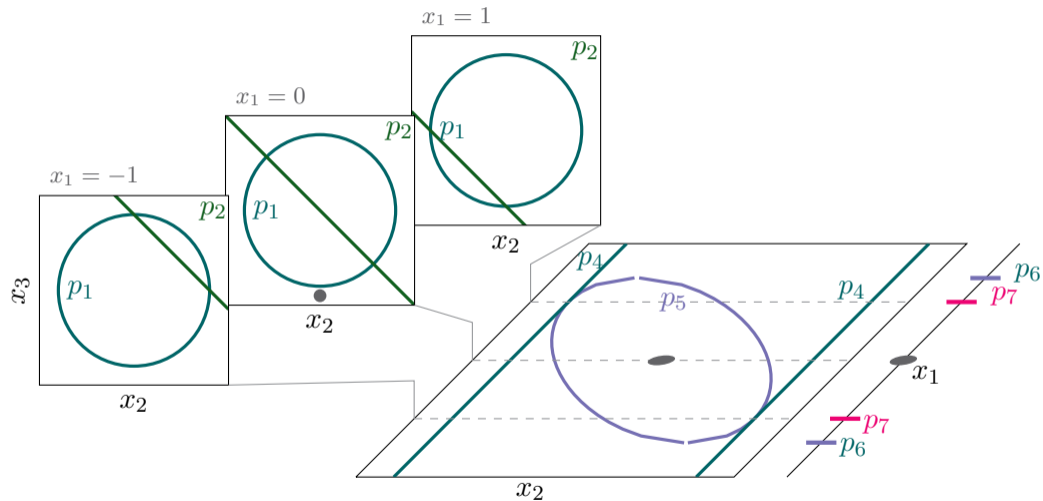
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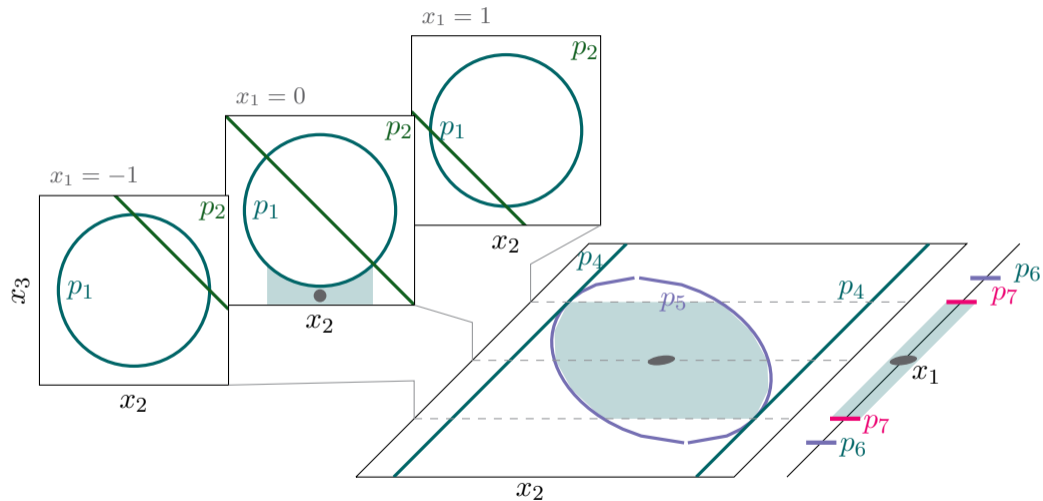
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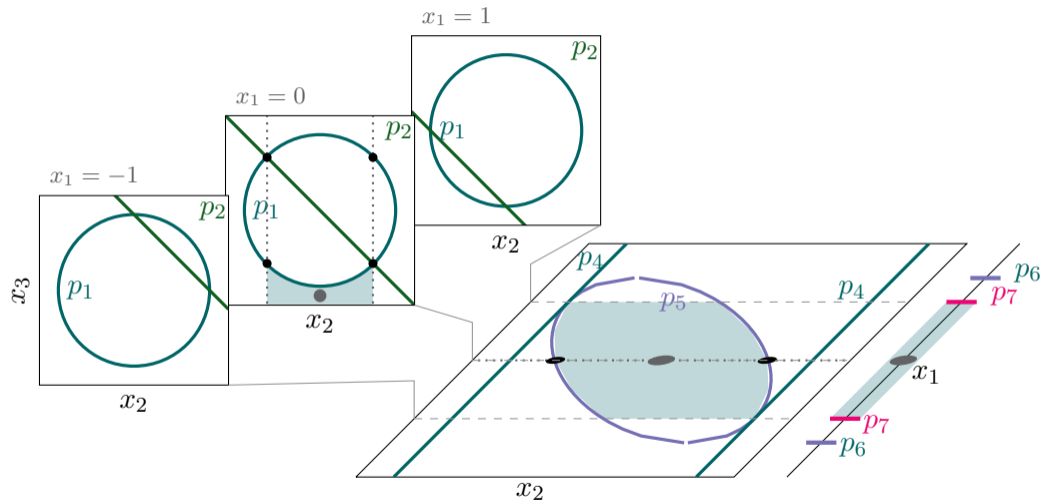
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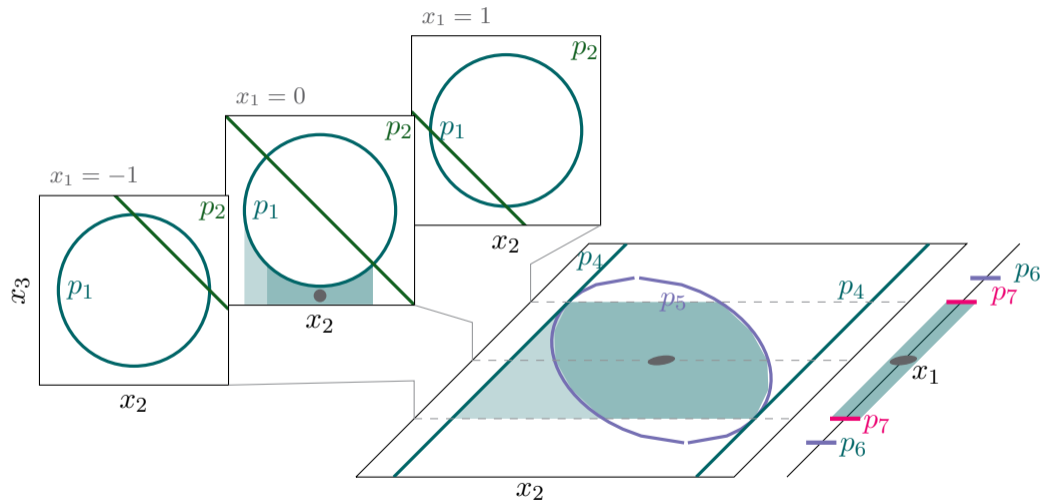
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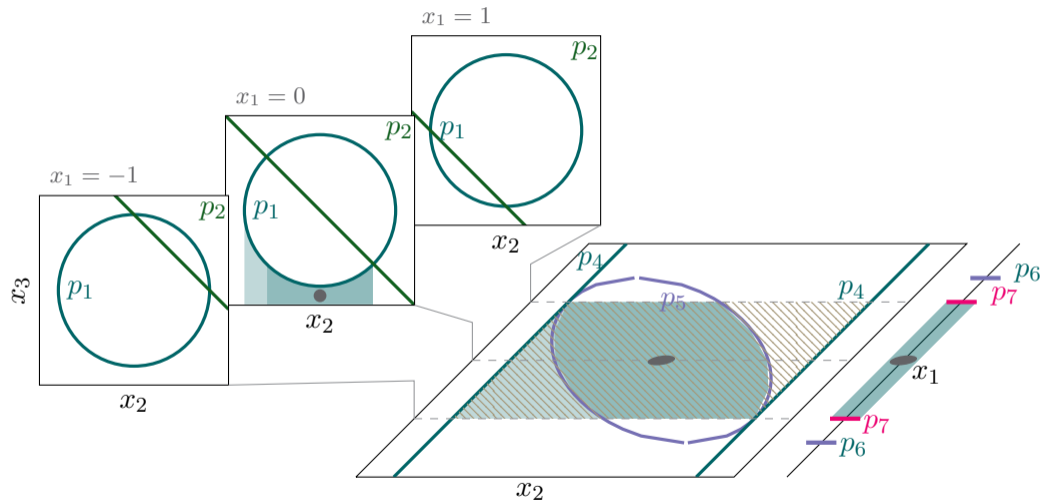
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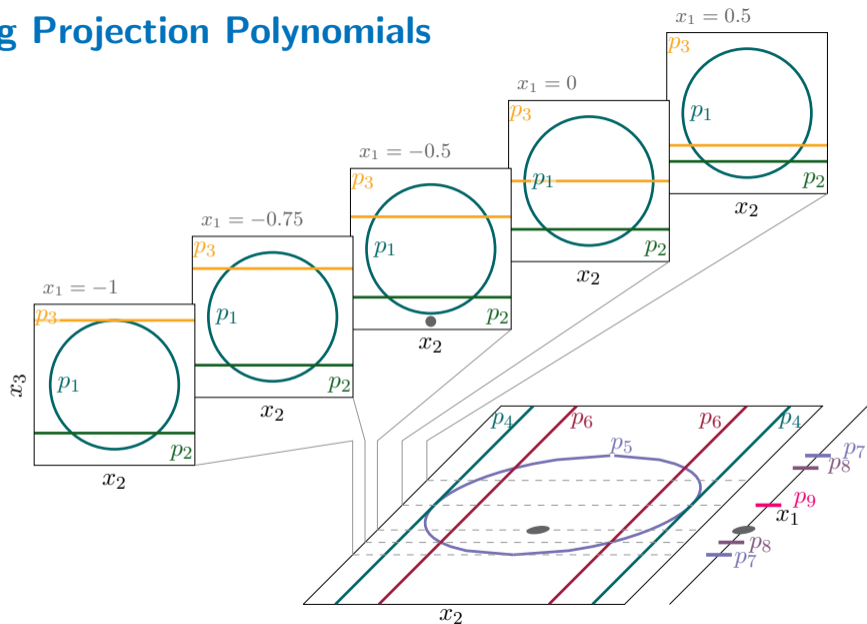
# Another Example



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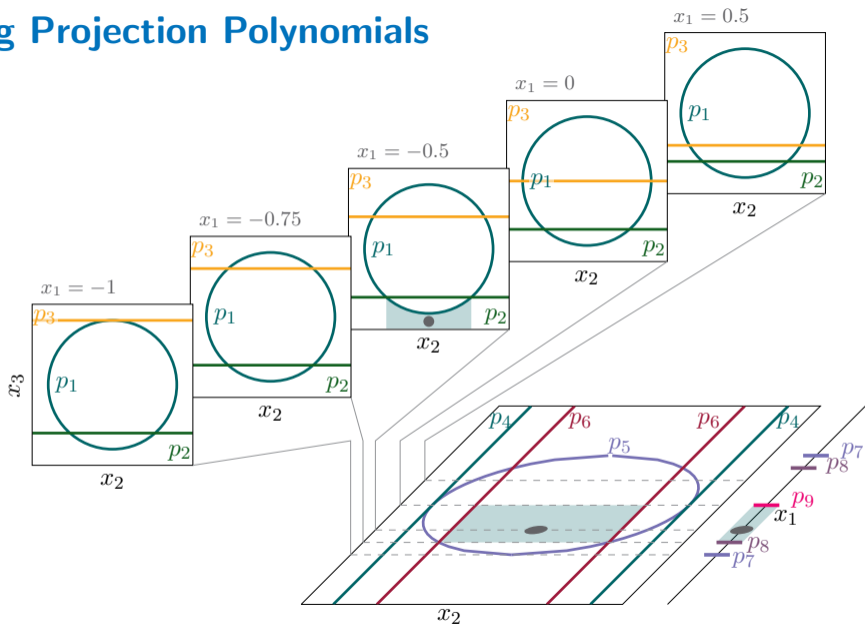


# Omitting Projection Polynomials

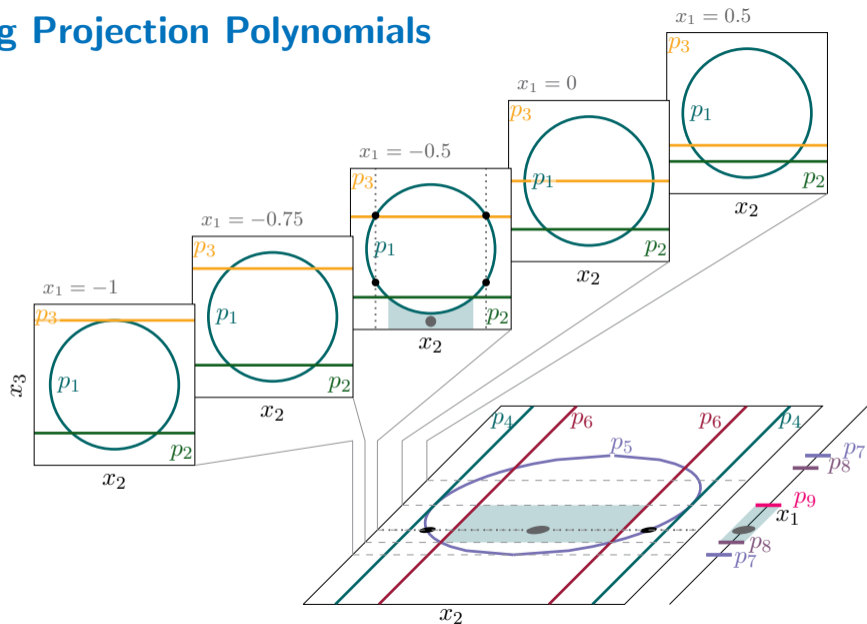




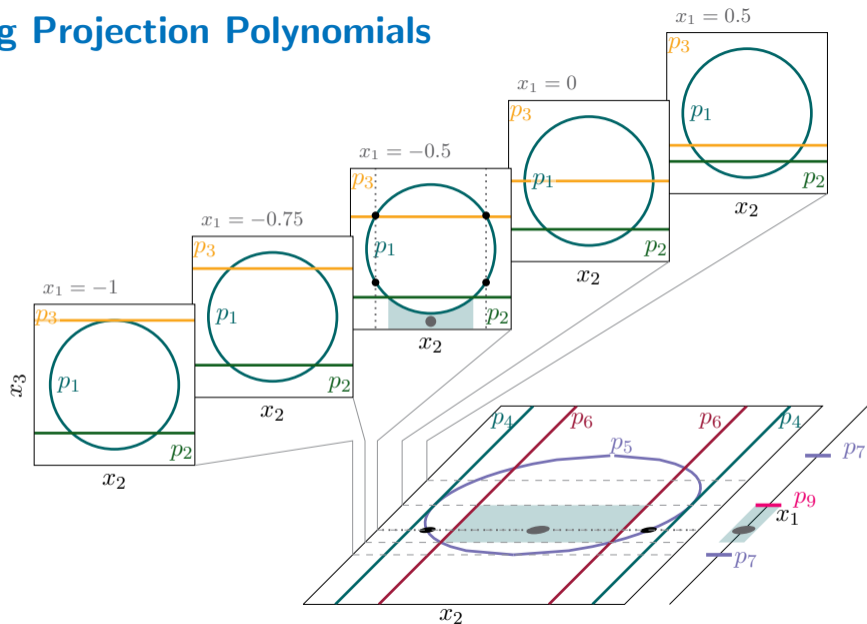
# Omitting Projection Polynomials



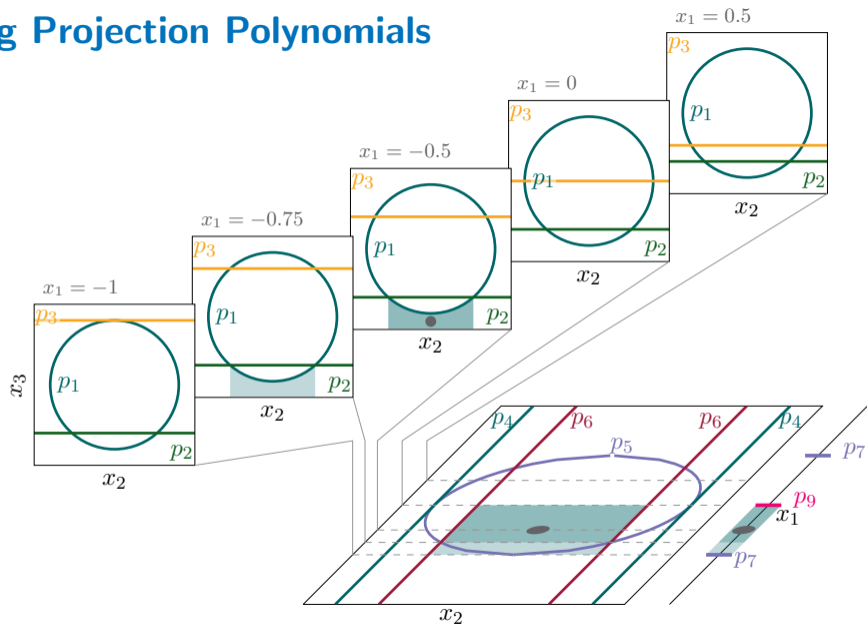
# Omitting Projection Polynomials



# Omitting Projection Polynomials



# Omitting Projection Polynomials



# Omitting Projection Polynomials

