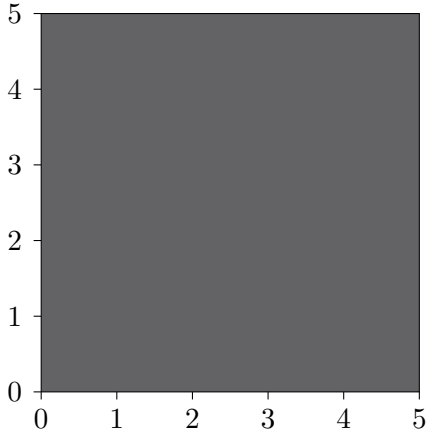
Cylindrical Algebraic Decomposition Based Methods in Satisfiability Modulo Non-linear Real Arithmetic

Jasper Nalbach

RWTH Aachen University

PhD Defence
November 12, 2025

What is Real Algebra?



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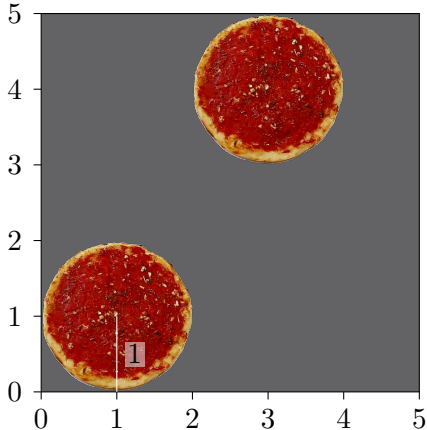

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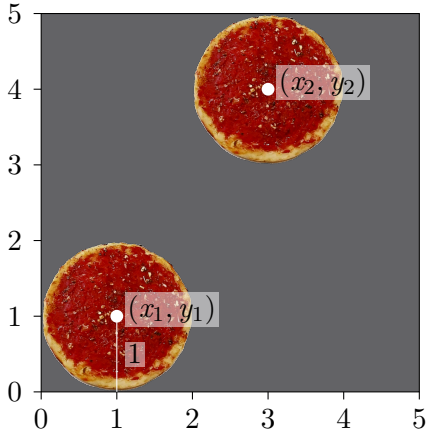

$$1 \leq x_1, x_2, y_1, y_2 \leq 5 - 1$$

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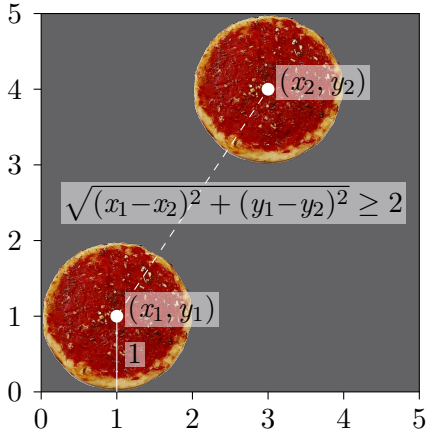

$$1 \leq x_1, x_2, y_1, y_2 \leq 5 - 1$$
$$\wedge\, (x_1 - x_2)^2 + (y_1 - y_2)^2 \geq 4$$

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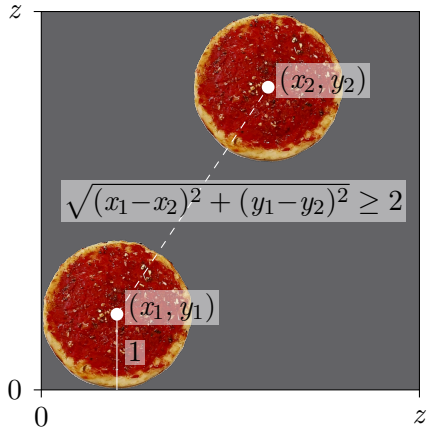

$$\forall z.\ 2 \leq z \leq 6 \rightarrow \exists x_1.\ \exists x_2.\ \exists y_1.\ \exists y_2.$$
$$1 \leq x_1, x_2, y_1, y_2 \leq z - 1$$
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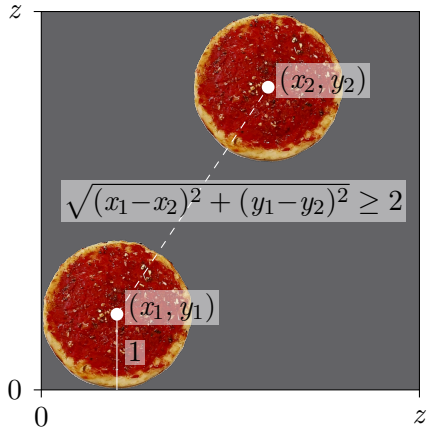

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$$\equiv z \geq 2 + \sqrt{2}$$

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History of Real Algebra

Al-Khwarizmi 820 al-Kitāb al-Mukhtaṣar fī Ḥisāb al-Jabr wal-Muqābalah (The Compendious Book on Calculation by Completion and Balancing)

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Cylindrical Algebraic Decomposition (Collins 1975) $\mathcal{O}(2^{2^n})$

Cylindrical Algebraic Decomposition [Collins 1975]

$$\exists x_1.\ \exists x_2.\ x_1^2 + x_2^2 - 1 < 0 \wedge x_1 + x_2 > 0$$

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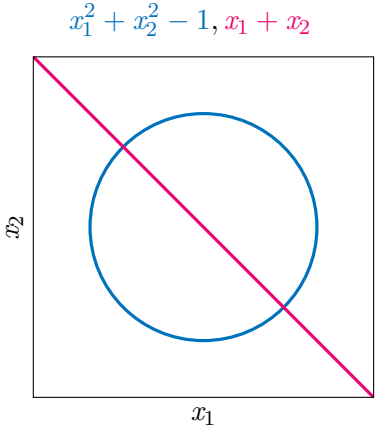

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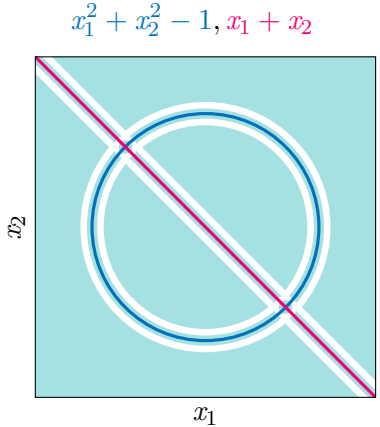

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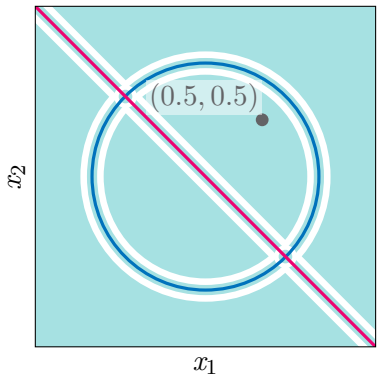

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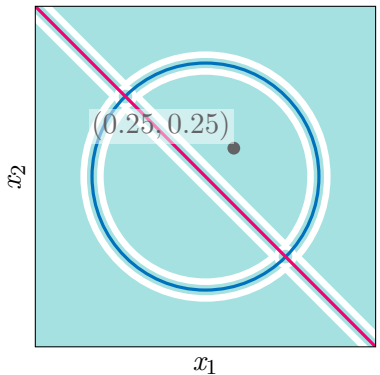

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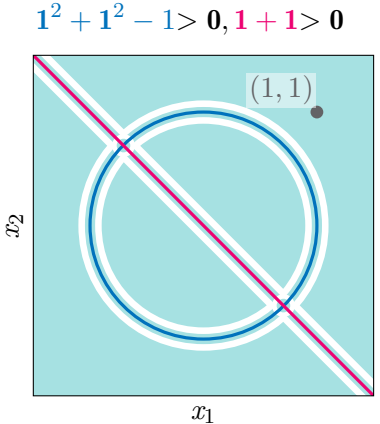

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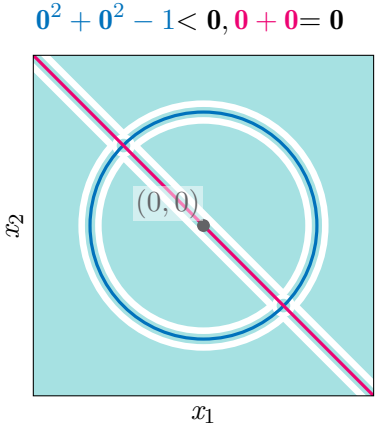

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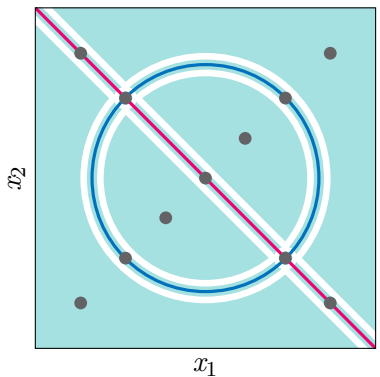

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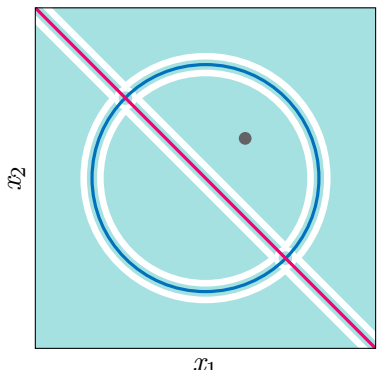

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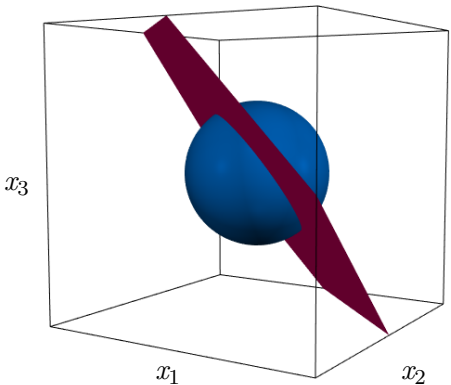

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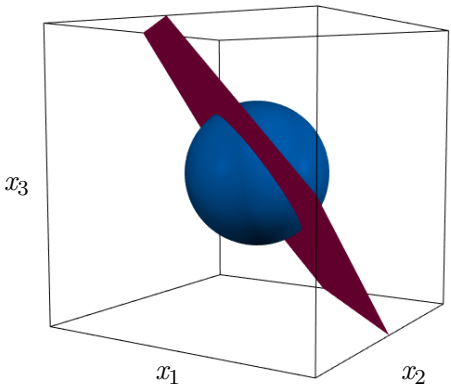

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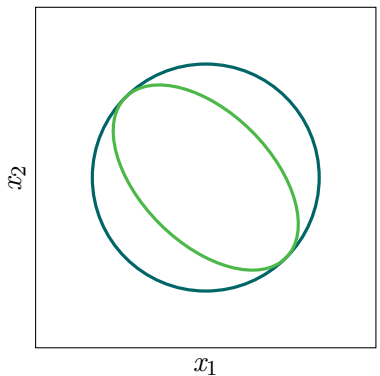

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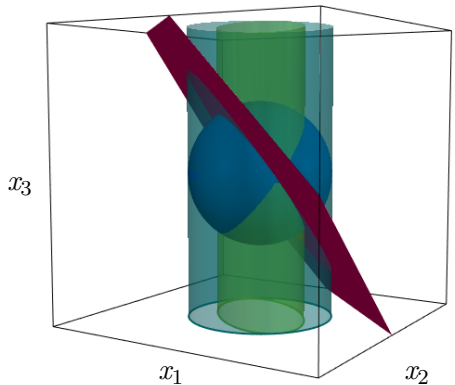

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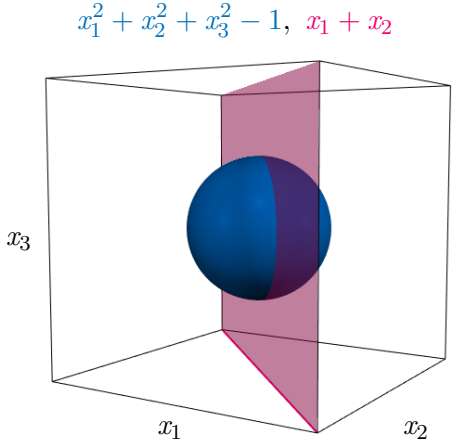

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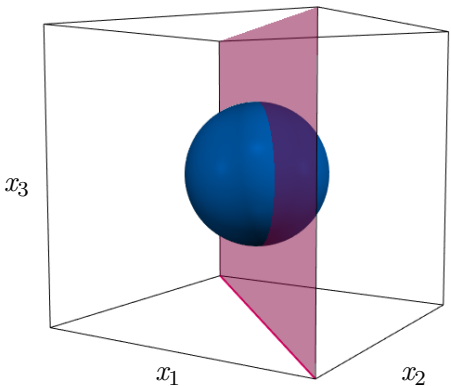

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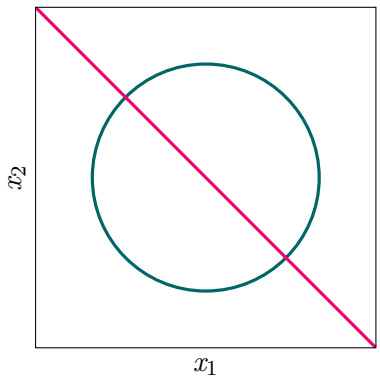

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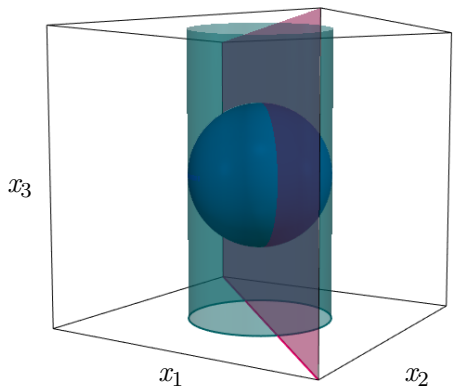

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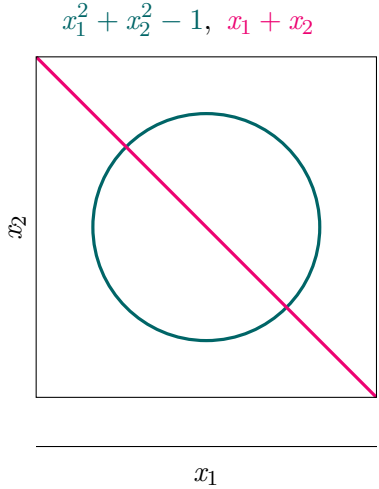

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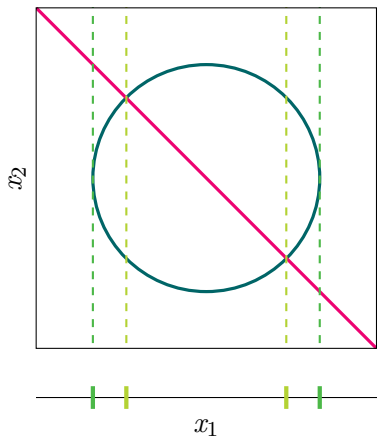

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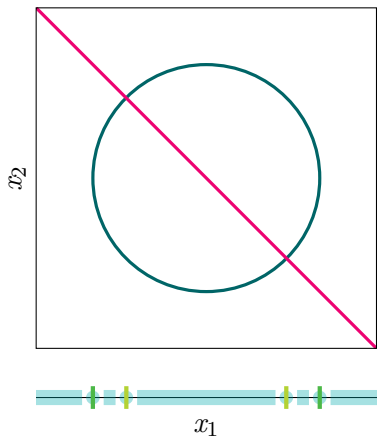

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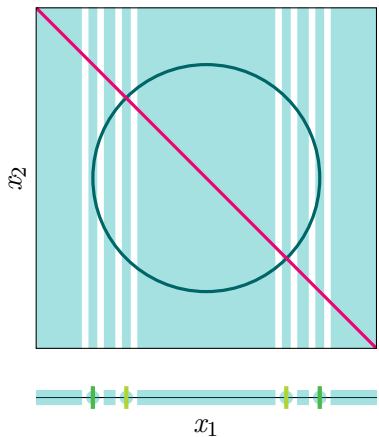

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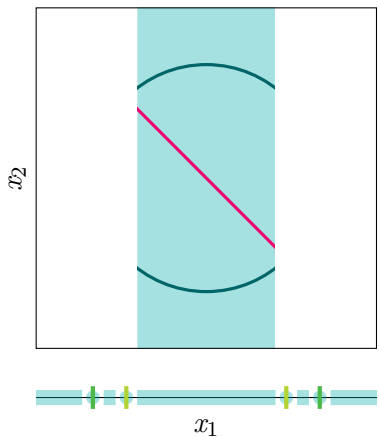

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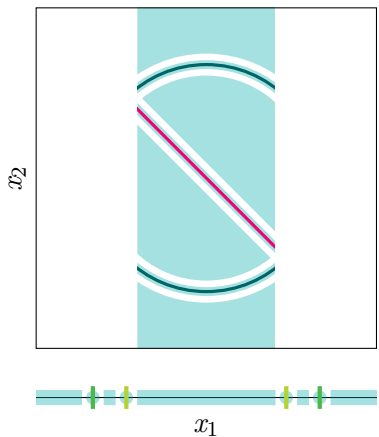

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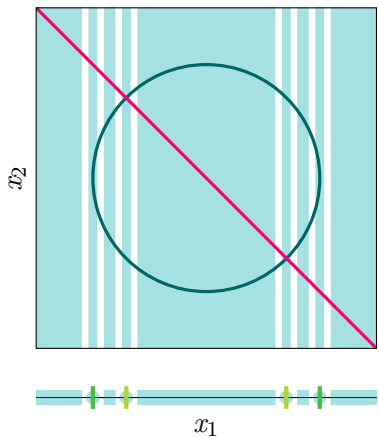

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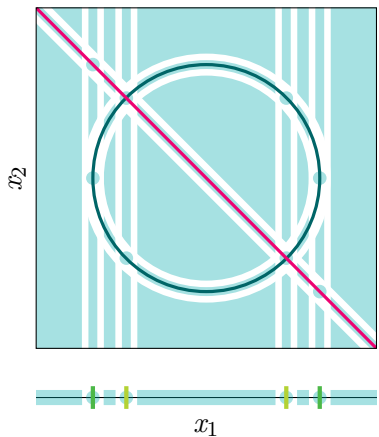

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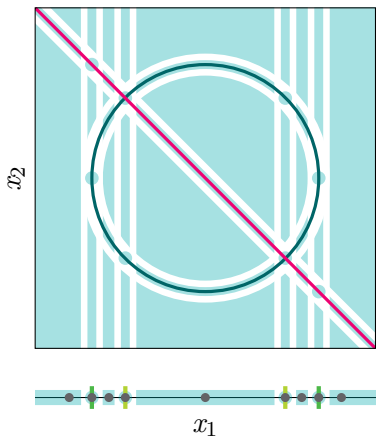

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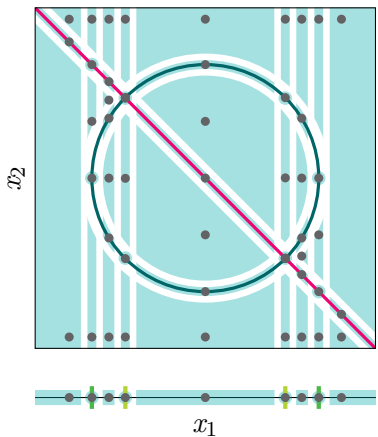

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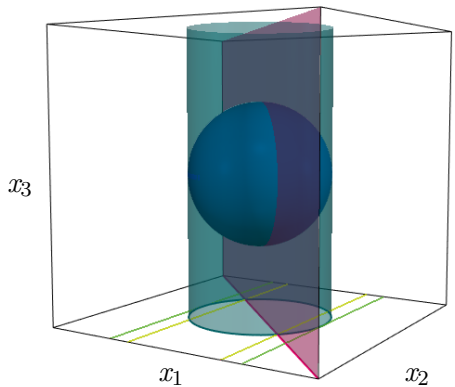

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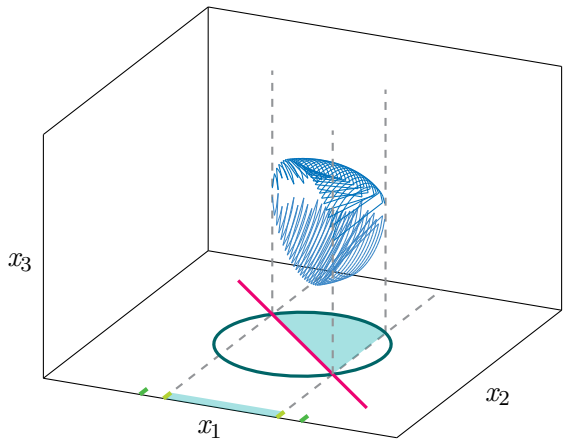

Limitations of CAD

Cylindrical Algebraic Decomposition (Collins 1975) First tractable approach. Doubly exponential in the number of variables.

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2^n

2^{2^n}

Image: Terezo del Río

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Basu, Pollack, Roy 1996 Singly exponential in the number of variables. Doubly exponential in the number of quantifier alternations.

Why use CAD at all then?

Hong 1991 *Comparison of Several Decision Algorithms for the Existential Theory of the Reals.*

Algorithm	Theoretical	$n = m = d = L = 2$
Collins	$L^3(md)^{2^{O(n)}}$	$\ll 1$ second
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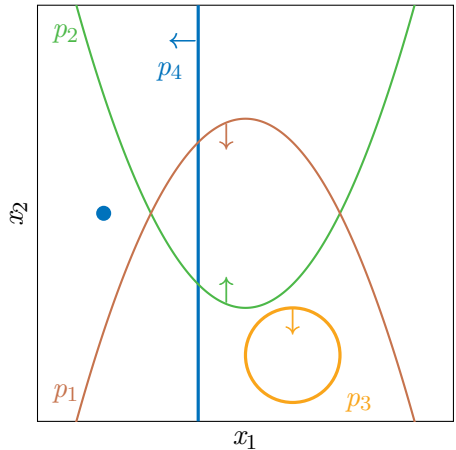
CAD solves real-world problems!

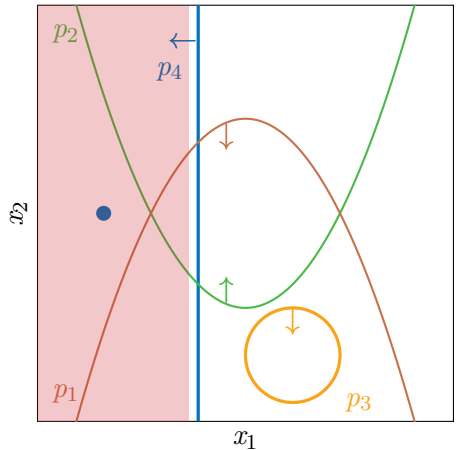
Exploration-guided Algorithms

$$\exists x_1.\exists x_2.\ \ p_1 < 0 \wedge p_2 > 0 \wedge p_3 \geq 0 \wedge p_4 \geq 0$$



Exploration-guided Algorithms

$$\exists x_1.\exists x_2.\ \ p_1 < 0 \wedge p_2 > 0 \wedge p_3 \geq 0 \wedge p_4 \geq 0$$



Exploration-guided Algorithms

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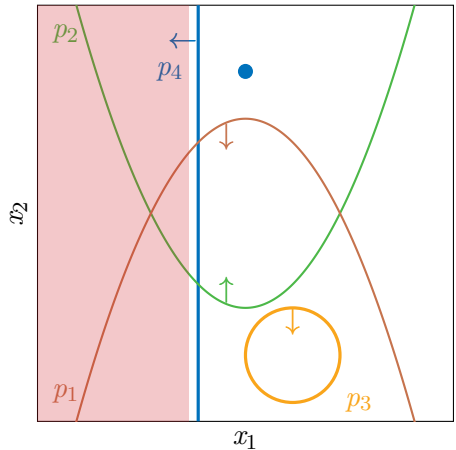
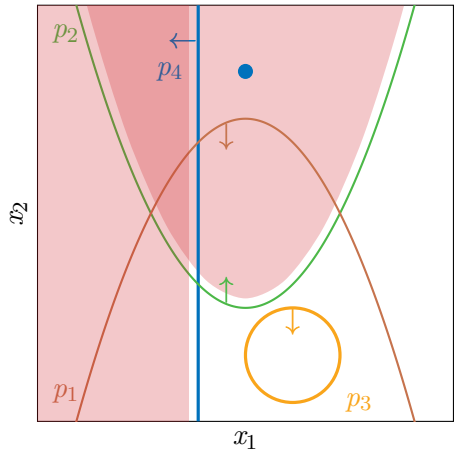


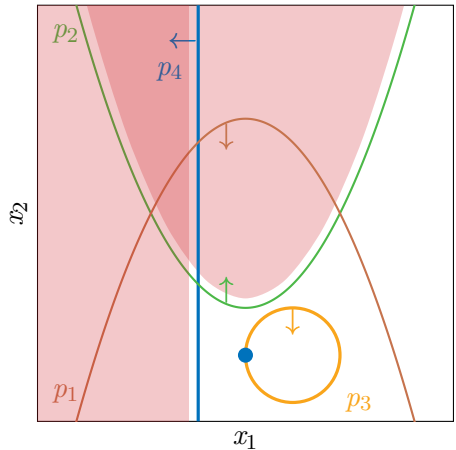
Exploration-guided Algorithms

$$\exists x_1.\exists x_2.\ \ p_1 < 0 \wedge p_2 > 0 \wedge p_3 \geq 0 \wedge p_4 \geq 0$$



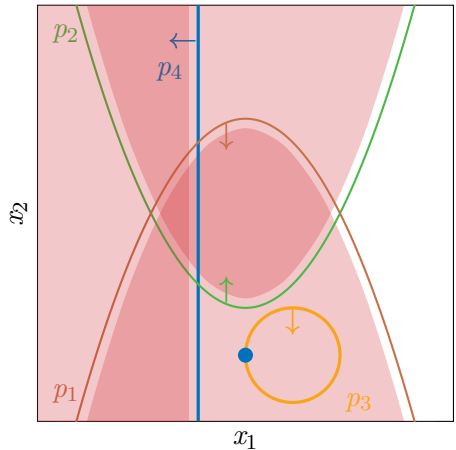
Exploration-guided Algorithms

$$\exists x_1.\exists x_2.\quad p_1 < 0 \wedge p_2 > 0 \wedge p_3 \geq 0 \wedge p_4 \geq 0$$



Exploration-guided Algorithms

$$\exists x_1.\exists x_2.\quad p_1 < 0 \wedge p_2 > 0 \wedge p_3 \geq 0 \wedge p_4 \geq 0$$



Exploration-guided Algorithms

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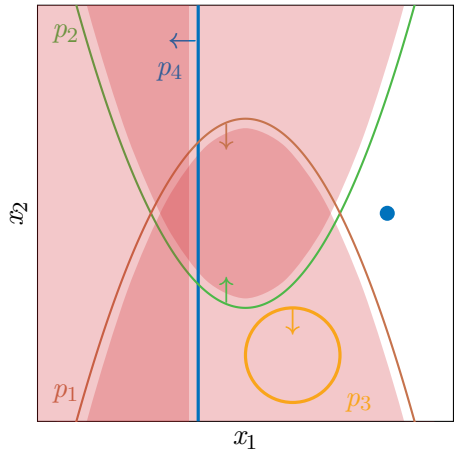
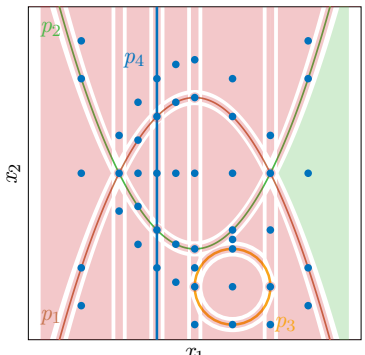
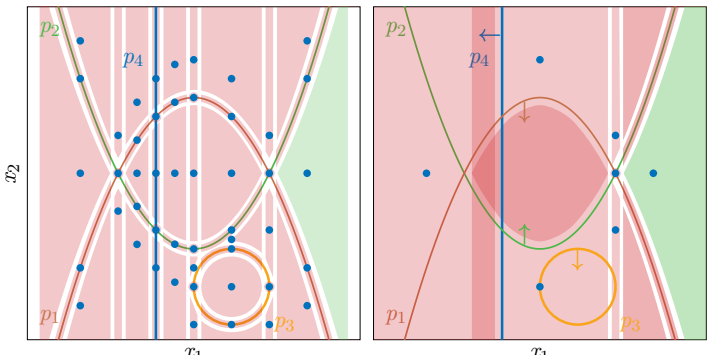


Exploration-guided Algorithms



Cylindrical Algebraic Decomposition (CAD) [Collins 1975]

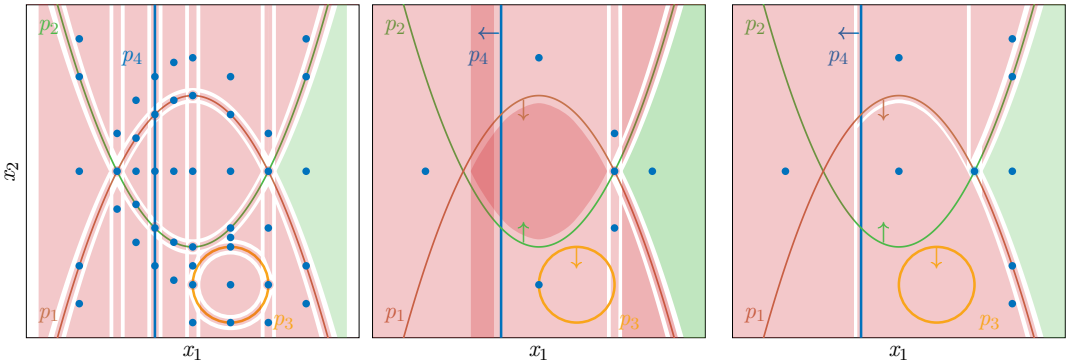
Exploration-guided Algorithms



Cylindrical Algebraic Decomposition (CAD) [Collins 1975]

NLSAT [Jovanović, de Moura 2012], Cylindrical Algebraic Covering (CAIC) [Ábrahám et al. 2021]

Exploration-guided Algorithms

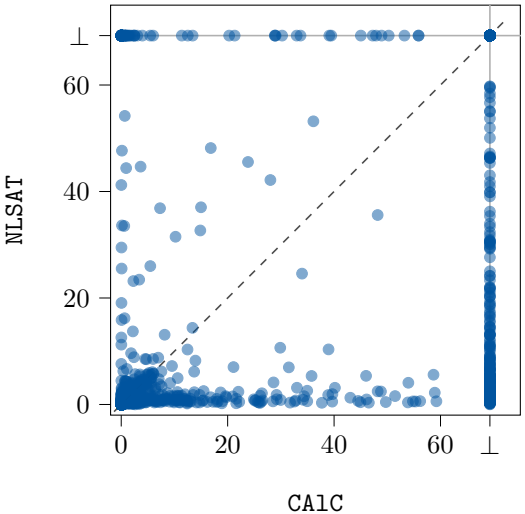


Cylindrical Algebraic Decomposition (CAD) [Collins 1975]

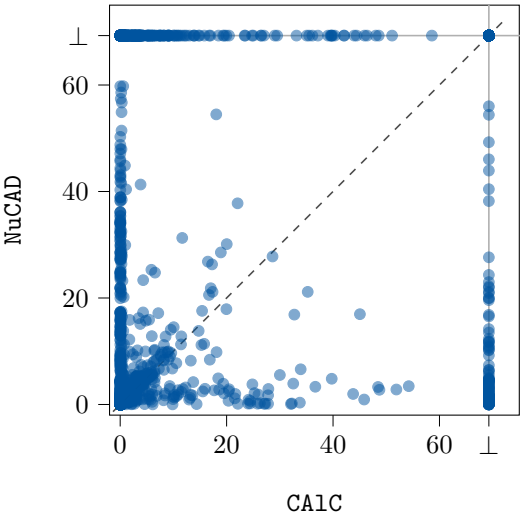
NLSAT [Jovanović, de Moura 2012], Cylindrical Algebraic Covering (CAIC) [Ábrahám et al. 2021]

Non-uniform CAD (NuCAD) [Brown 2015]

Comparison of Running Times

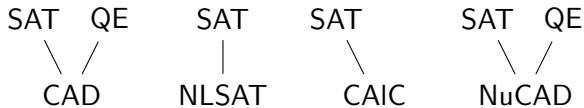


Satisfiability Checking



Quantifier Elimination

Contributions



Contributions

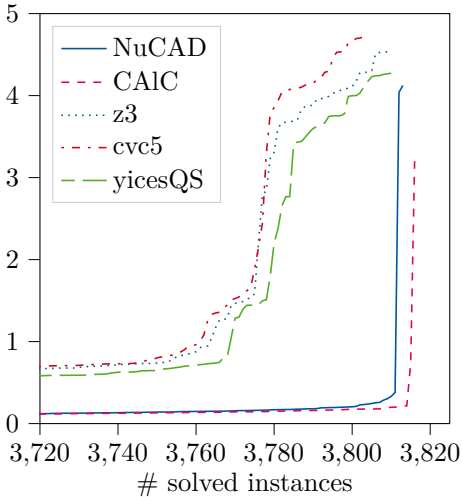
Quantifier Elimination

$$\exists x_1.\ \exists x_2.\ \exists y_1.\ \exists y_2.\ 1 \leq x_1, x_2, y_1, y_2 \leq z - 1 \wedge (x_1 - x_2)^2 + (y_1 - y_2)^2 \geq 4$$

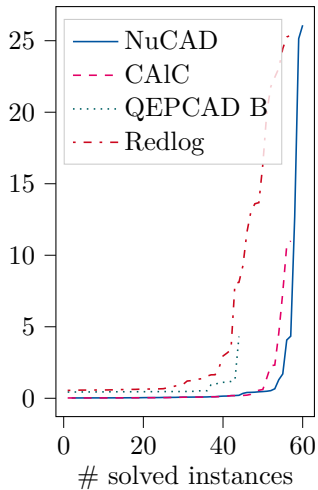
Quantifier Elimination

$$\exists x_1.\ \exists x_2.\ \exists y_1.\ \exists y_2.\ 1 \leq x_1, x_2, y_1, y_2 \leq z - 1 \wedge (x_1 - x_2)^2 + (y_1 - y_2)^2 \geq 4$$
$$\equiv z \geq 2 + \sqrt{2}$$

Quantifier Elimination [Nalbach, Kremer 2024; Nalbach, Ábrahám 2025]

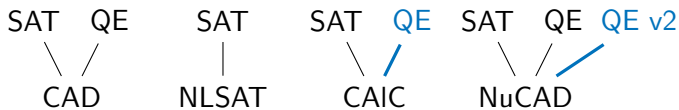


Satisfiability Checking with Quantifiers

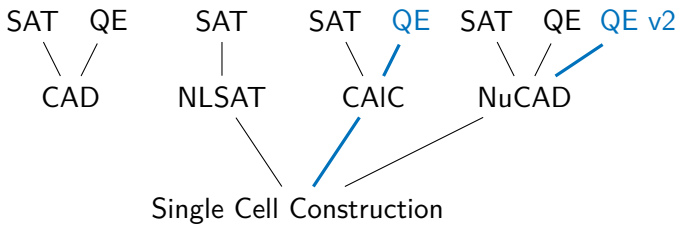


Quantifier Elimination

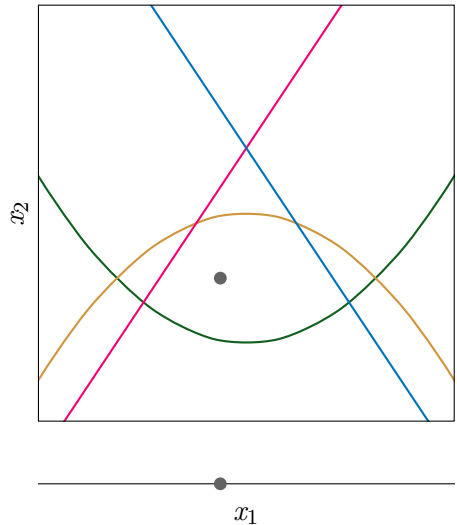
Contributions



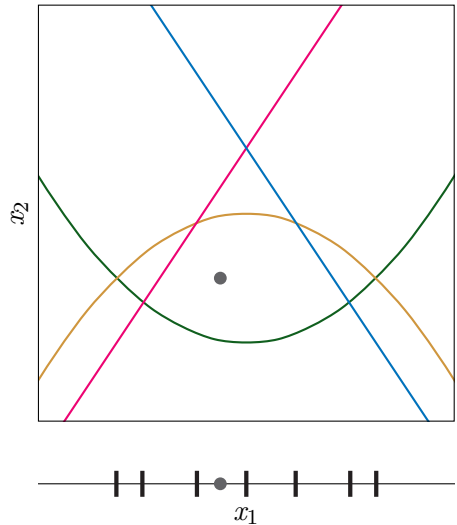
Contributions



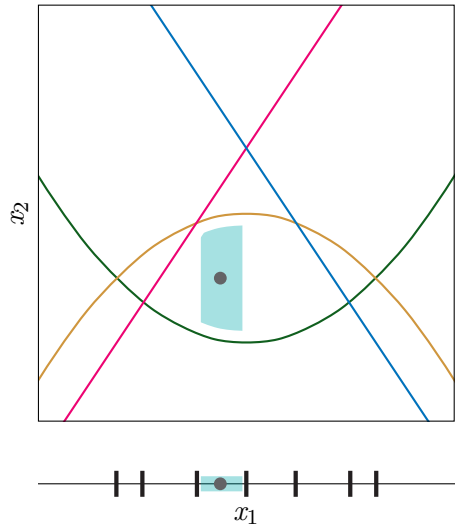
Single Cell Construction [Brown 2015]



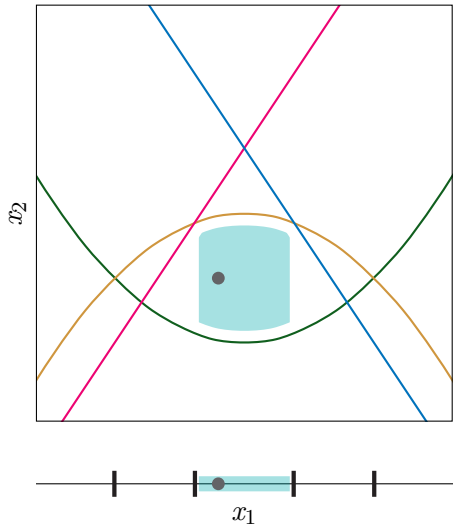
Single Cell Construction [Brown 2015]



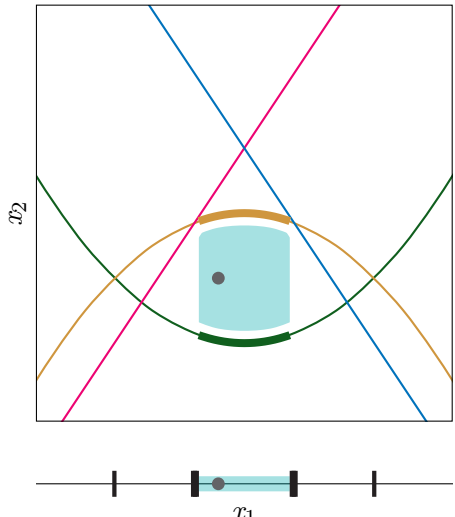
Single Cell Construction [Brown 2015]



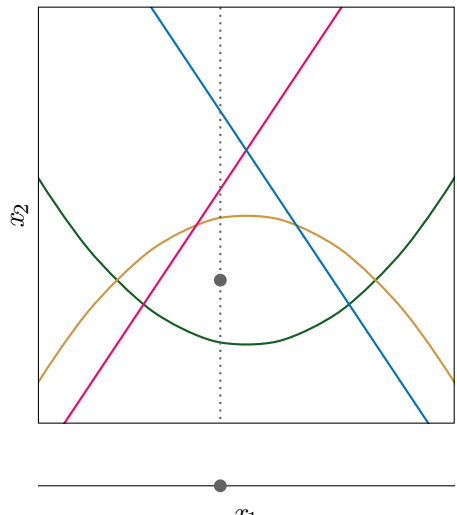
Single Cell Construction [Brown 2015]



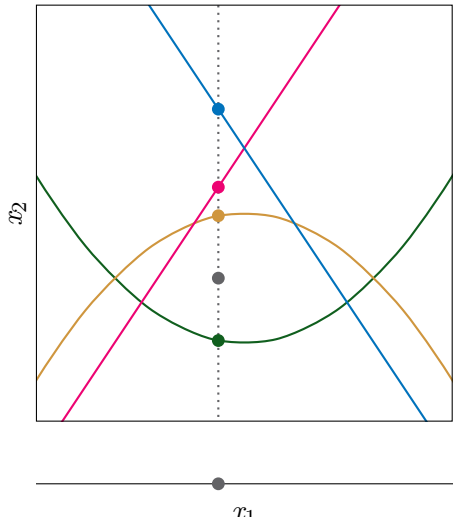
Single Cell Construction [Brown 2015]



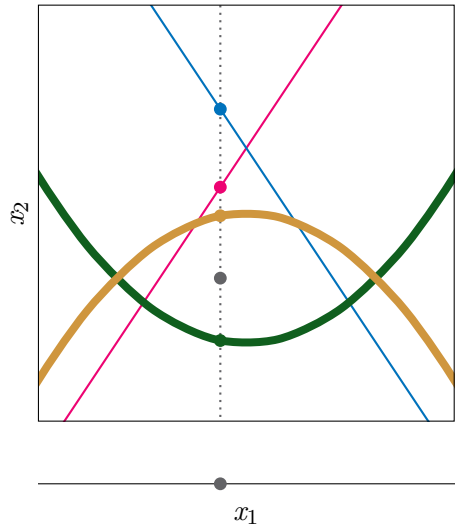
Single Cell Construction [Brown 2015, Nalbach et al. 2024]



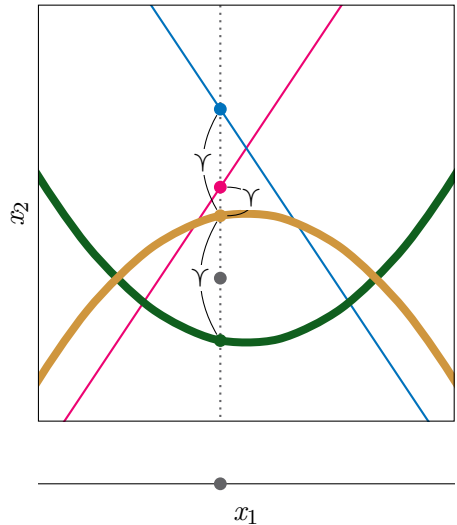
Single Cell Construction [Brown 2015, Nalbach et al. 2024]



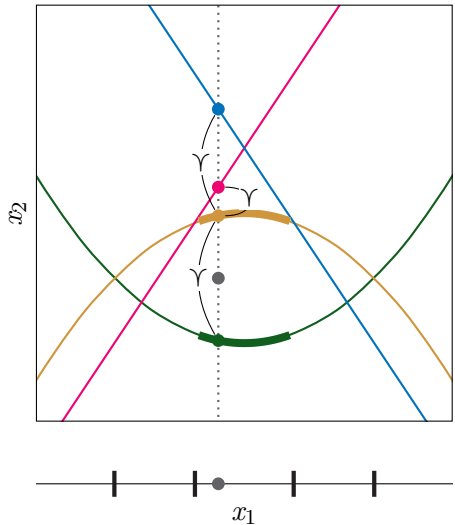
Single Cell Construction [Brown 2015, Nalbach et al. 2024]



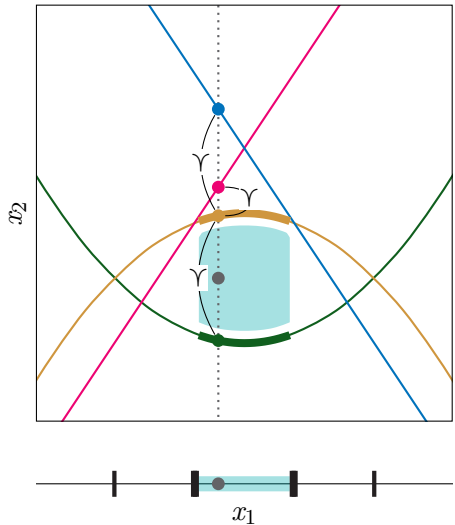
Single Cell Construction [Brown 2015, Nalbach et al. 2024]



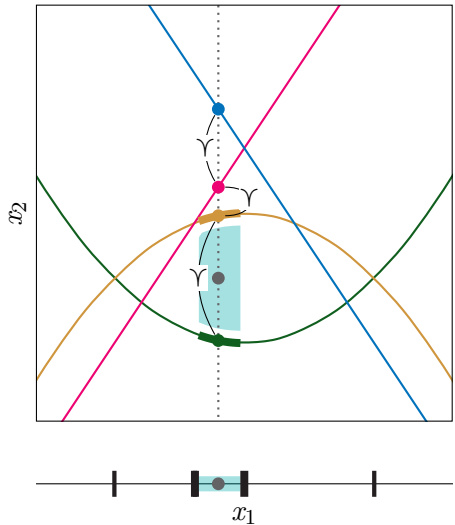
Single Cell Construction [Brown 2015, Nalbach et al. 2024]



Single Cell Construction [Brown 2015, Nalbach et al. 2024]



Single Cell Construction [Brown 2015, Nalbach et al. 2024]

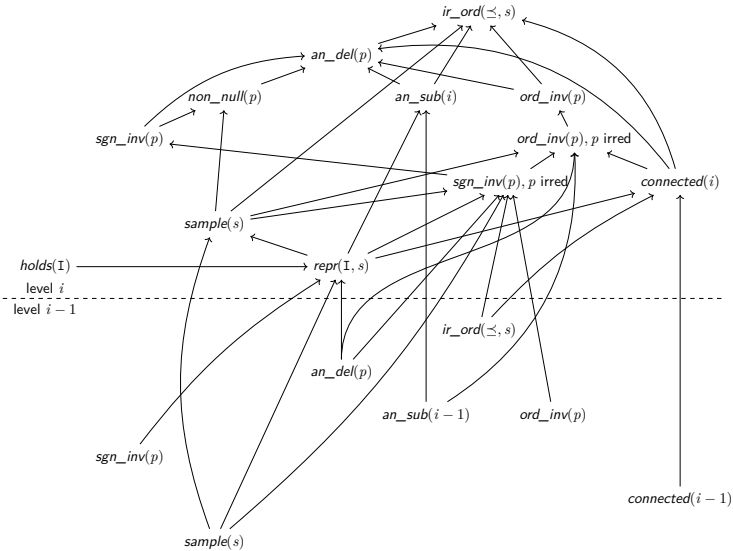


A Proof System for Exploration-Guided Algorithms

Rule. Let $i \in \mathbb{N}_{>0}$, $R \subseteq \mathbb{R}^i$, $p \in \mathbb{Q}[x_1, \ldots, x_i]$, $\mathsf{level}(p) = i$, p reducible, $\mathsf{factors}(p) = \{q_1, \ldots, q_k\}$.

$$\mathsf{sgn\_inv}(q_1)(R),\ \ldots,\ \mathsf{sgn\_inv}(q_k)(R) \qquad \vdash \mathsf{sgn\_inv}(p)(R)$$

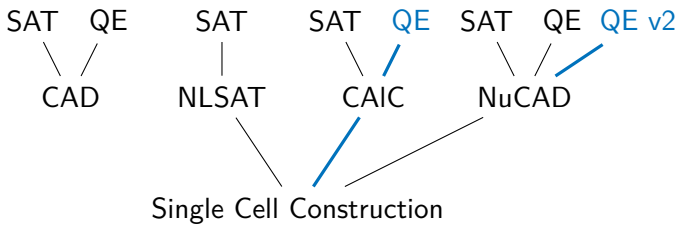
A Proof System for Exploration-Guided Algorithms



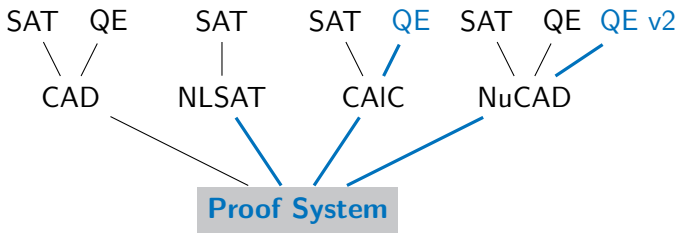
A Proof System for Exploration-Guided Algorithms

$$
\begin{aligned}
\boxed{\mathit{sgn\_inv}(p_1)} &\to \mathit{sample}(s), \mathit{repr}(\mathtt{I}_2, s_1), \mathit{ir\_ord}(\preceq, s_1), \mathit{an\_del}(p_1), \mathit{an\_sub}(1), \mathit{connected}(1)\\
\boxed{\mathit{sgn\_inv}(p_2)} &\to \mathit{sample}(s), \mathit{repr}(\mathtt{I}_2, s_1), \mathit{ir\_ord}(\preceq, s_1), \mathit{an\_del}(p_2), \mathit{an\_sub}(1), \mathit{connected}(1)\\
\boxed{\mathit{sgn\_inv}(p_3)} &\to \mathit{sample}(s), \mathit{repr}(\mathtt{I}_2, s_1), \mathit{ir\_ord}(\preceq, s_1), \mathit{an\_del}(p_3), \mathit{an\_sub}(1), \mathit{connected}(1)\\
\mathit{sample}(s) &\to \mathit{repr}(\mathtt{I}_2, s_1), \mathit{sample}(s_1)\\
\mathit{repr}(\mathtt{I}_2, s_1) &\to \boxed{R = R{\downarrow_{[1]}} \times \mathtt{I}_2}, \mathit{an\_del}(p_2), \mathit{an\_del}(p_3), \mathit{sample}(s_1)\\
\mathit{ir\_ord}(\preceq, s_1) &\to \mathit{an\_del}(p_1), \mathit{an\_del}(p_2), \mathit{an\_del}(p_3), \mathit{ord\_inv}(\mathrm{res}_{x_2}(p_3, p_1)), \mathit{ord\_inv}(\mathrm{res}_{x_2}(p_3, p_2)),\\
&\qquad \mathit{an\_sub}(1), \mathit{connected}(1), \mathit{sample}(s_1)\\
\mathit{an\_del}(p_1) &\to \mathit{non\_null}(p_1), \mathit{ord\_inv}(\mathrm{disc}_{x_2}(p_1)), \mathit{an\_sub}(1), \mathit{connected}(1), \mathit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_1))\\
\mathit{an\_del}(p_2) &\to \mathit{non\_null}(p_2), \mathit{ord\_inv}(\mathrm{disc}_{x_2}(p_2)), \mathit{an\_sub}(1), \mathit{connected}(1), \mathit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_2))\\
\mathit{an\_del}(p_3) &\to \mathit{non\_null}(p_3), \mathit{ord\_inv}(\mathrm{disc}_{x_2}(p_3)), \mathit{an\_sub}(1), \mathit{connected}(1), \mathit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_3))\\
\mathit{non\_null}(p_1) &\to \mathit{trivial}\\
\mathit{non\_null}(p_2) &\to \mathit{trivial}\\
\mathit{non\_null}(p_3) &\to \mathit{trivial}\\
\mathit{ord\_inv}(\mathrm{disc}_{x_2}(p_1)) &\to \mathit{trivial}\\
\mathit{ord\_inv}(\mathrm{disc}_{x_2}(p_2)) &\to \mathit{sgn\_inv}(p_4), \mathit{sample}(s_1)\\
\mathit{ord\_inv}(\mathrm{disc}_{x_2}(p_3)) &\to \mathit{trivial}\\
\mathit{sgn\_inv}(p_4) &\to \mathit{repr}(\mathtt{I}_1)\\
\mathit{ord\_inv}(\mathrm{res}_{x_2}(p_3, p_1)) &\to \mathit{trivial}\\
\mathit{ord\_inv}(\mathrm{res}_{x_2}(p_3, p_2)) &\to \mathit{sgn\_inv}(p_5), \mathit{sample}(s_1)
\end{aligned}
\qquad
\begin{aligned}
\mathit{sgn\_inv}(p_5) &\to \mathit{repr}(\mathtt{I}_1)\\
\mathit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_1)) &\to \mathit{trivial}\\
\mathit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_2)) &\to \mathit{trivial}\\
\mathit{sgn\_inv}(\mathrm{ldcf}_{x_2}(p_3)) &\to \mathit{trivial}\\
\mathit{an\_sub}(1) &\to \mathit{repr}(\mathtt{I}_1)\\
\mathit{connected}(1) &\to \mathit{trivial}\\
\mathit{sample}(s_1) &\to \mathit{repr}(\mathtt{I}_1)\\
\mathit{repr}(\mathtt{I}_1) &\to \boxed{R{\downarrow_{[1]}} = \mathtt{I}_1}
\end{aligned}
$$

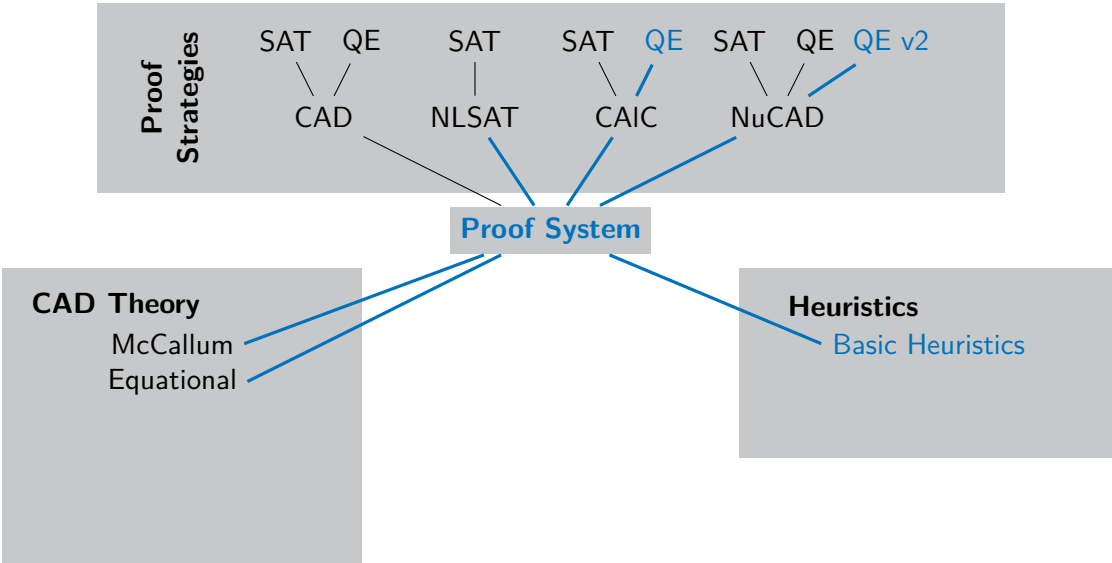
Contributions



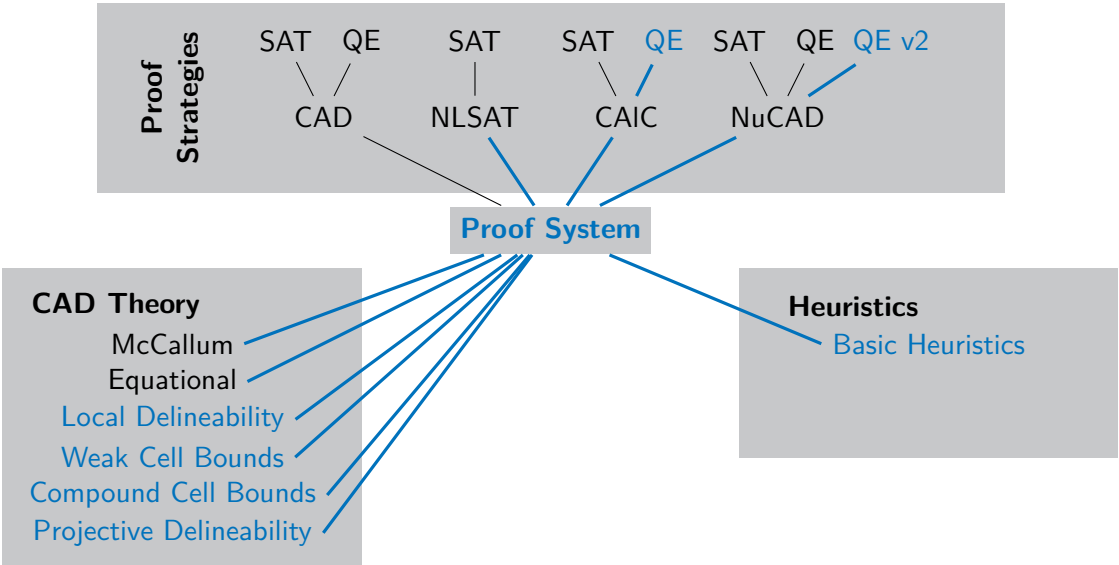
Contributions



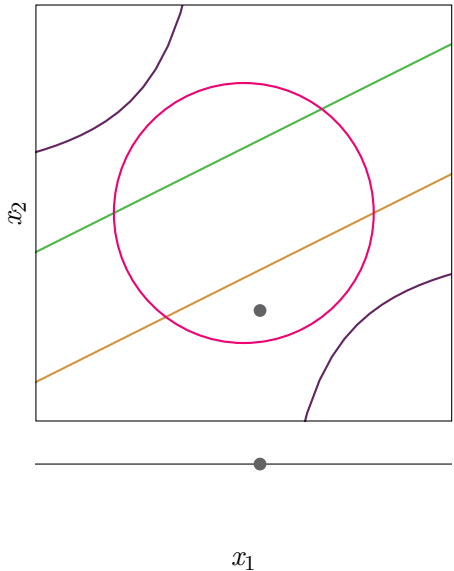
Contributions



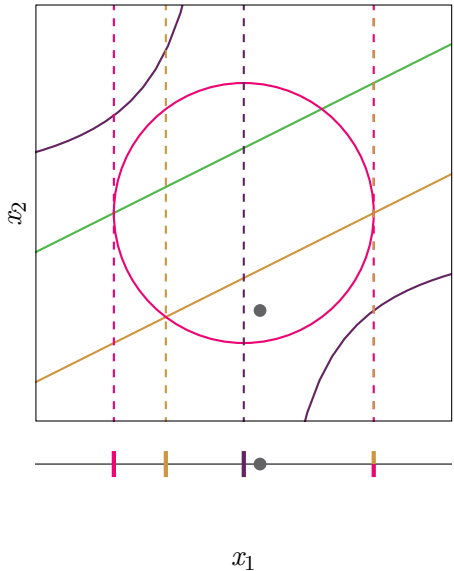
Contributions



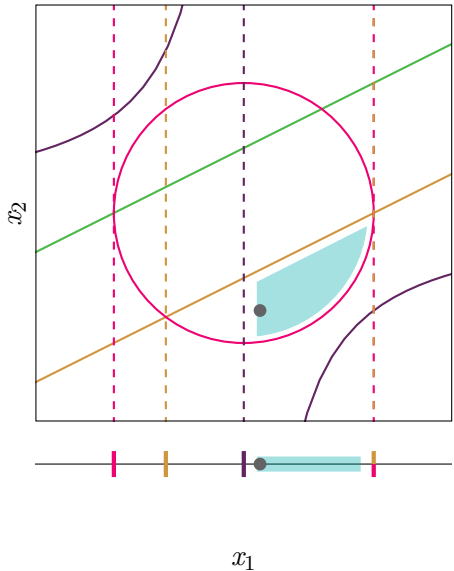
Projective Delineability [Michel et al. 2024, Nalbach et al. 2025]



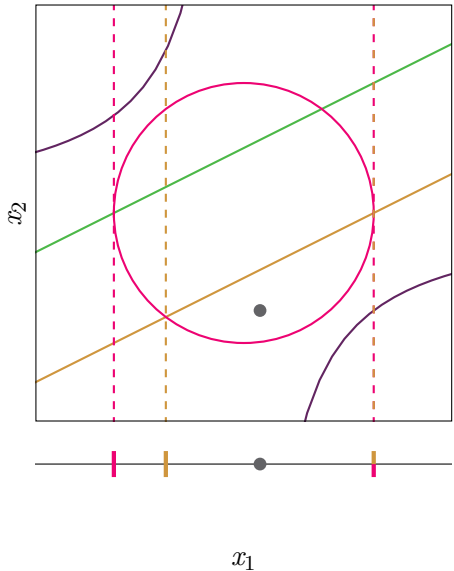
Projective Delineability [Michel et al. 2024, Nalbach et al. 2025]

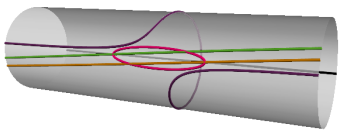


Projective Delineability [Michel et al. 2024, Nalbach et al. 2025]

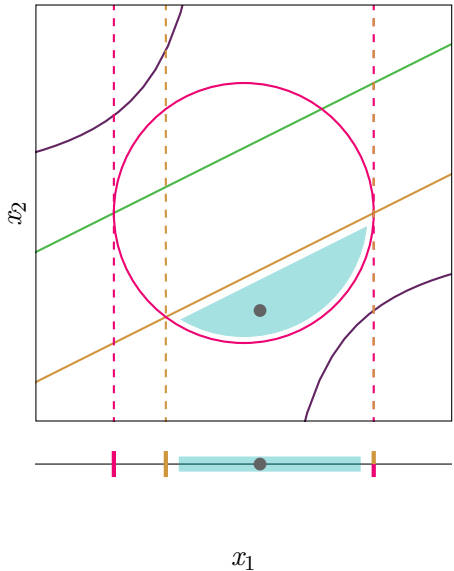


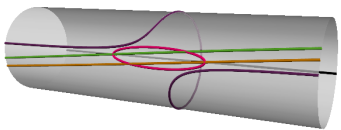
Projective Delineability [Michel et al. 2024, Nalbach et al. 2025]



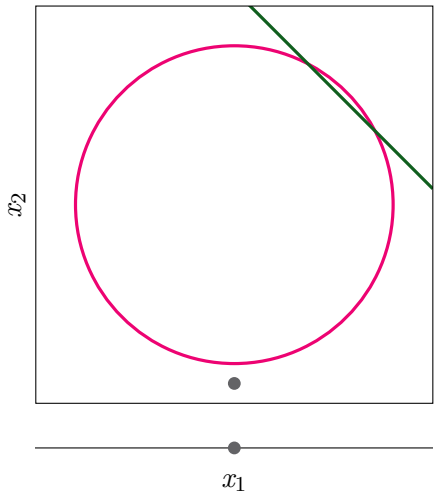


Projective Delineability [Michel et al. 2024, Nalbach et al. 2025]

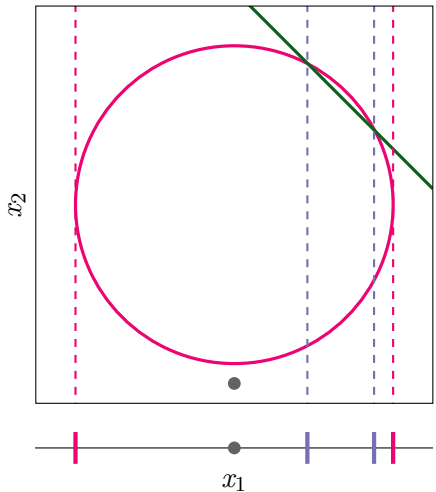




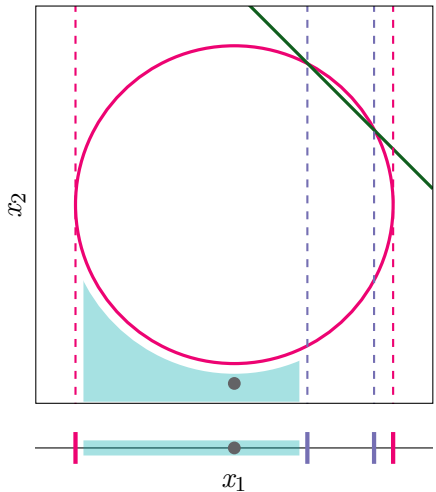
Irrelevant Intersections [Nalbach, Ábrahám 2024]



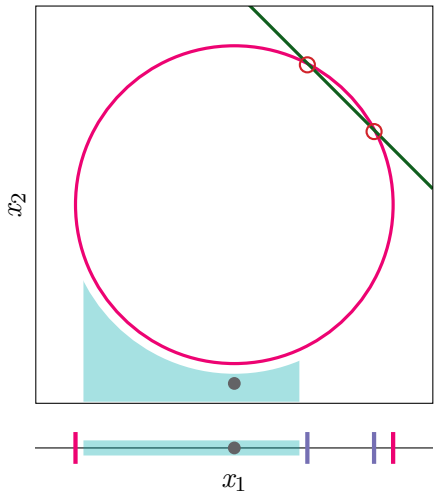
Irrelevant Intersections [Nalbach, Ábrahám 2024]



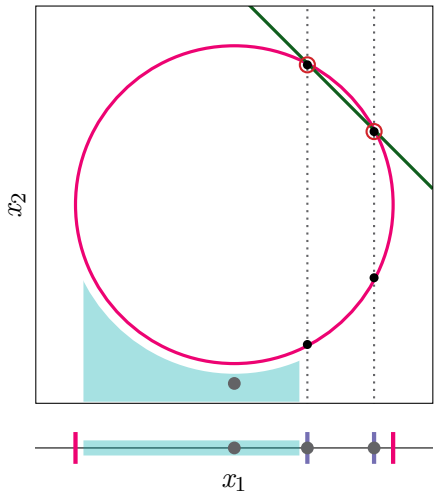
Irrelevant Intersections [Nalbach, Ábrahám 2024]



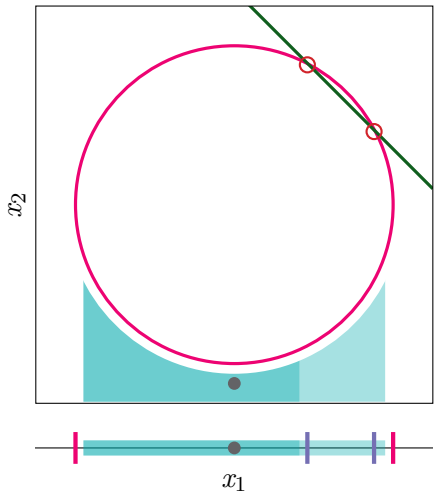
Irrelevant Intersections [Nalbach, Ábrahám 2024]



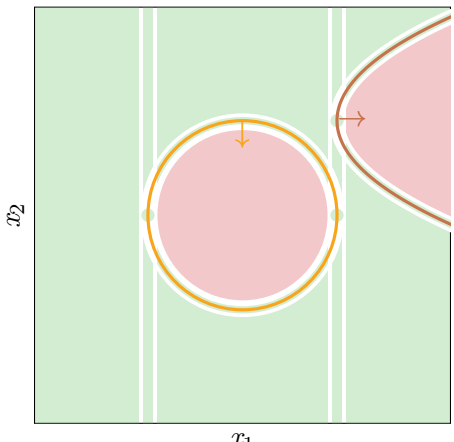
Irrelevant Intersections [Nalbach, Ábrahám 2024]



Irrelevant Intersections [Nalbach, Ábrahám 2024]

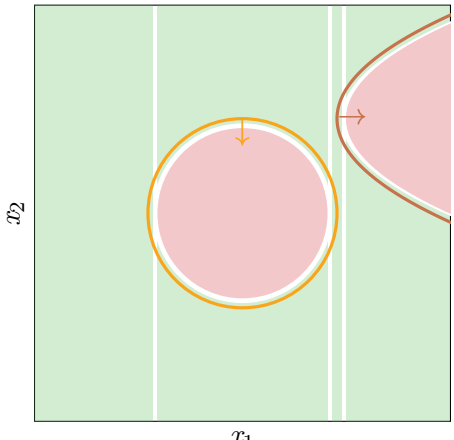


Weak Cell Bounds [Nalbach, Ábrahám 2024; Nalbach, Ábrahám 2025]



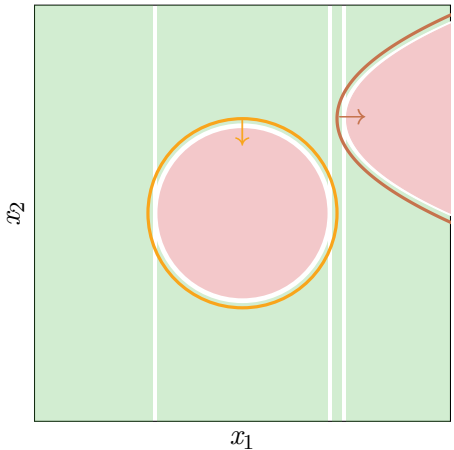
NuCAD Example

Weak Cell Bounds [Nalbach, Ábrahám 2024; Nalbach, Ábrahám 2025]

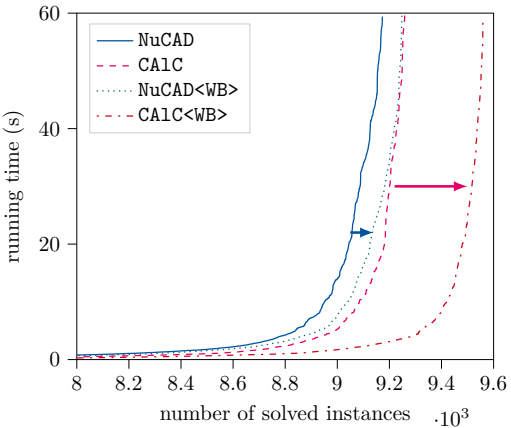


NuCAD Example
from 19 to 8 cells

Weak Cell Bounds [Nalbach, Ábrahám 2024; Nalbach, Ábrahám 2025]

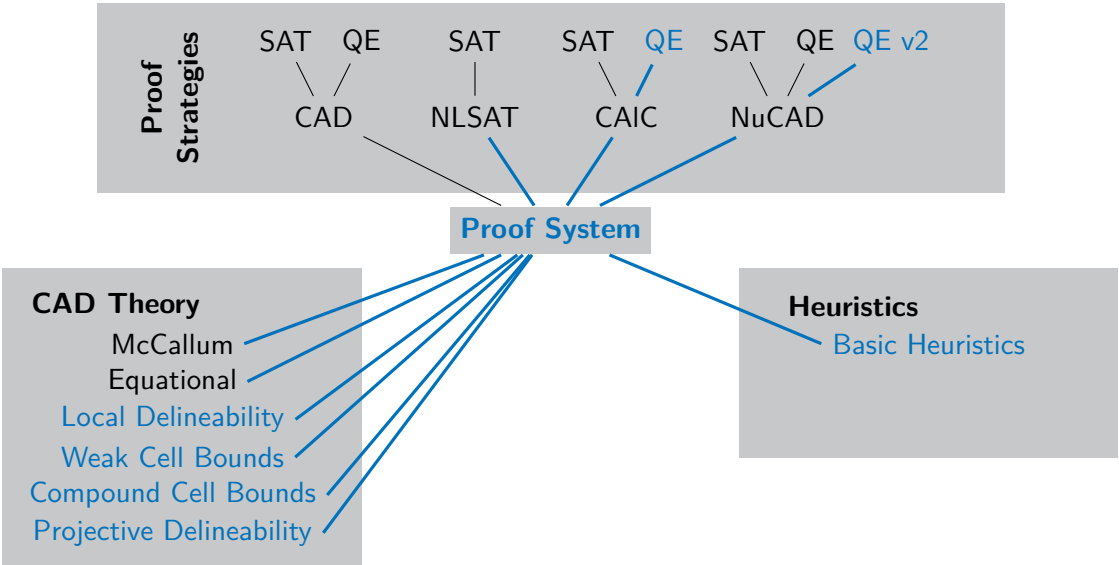


NuCAD Example
from 19 to 8 cells

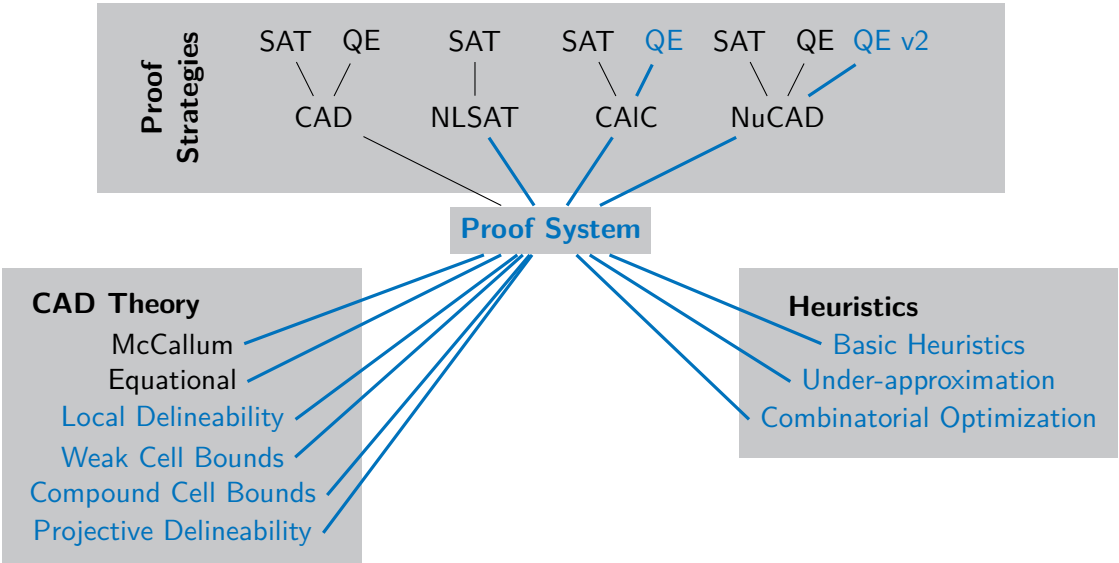


Quantifier Elimination

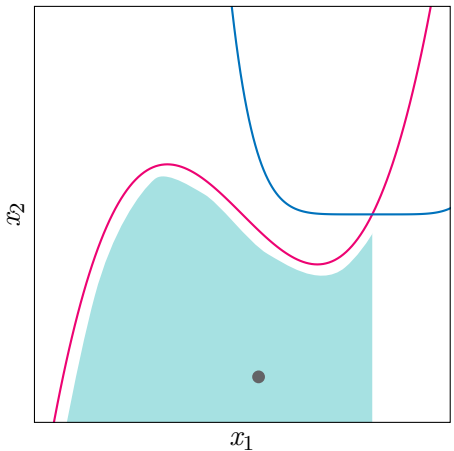
Contributions



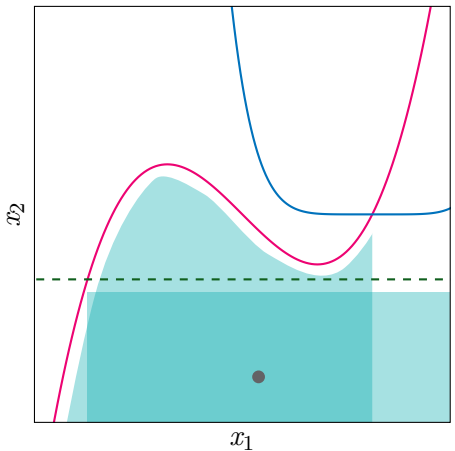
Contributions



Under-approximation of Cell Bounds [Promies et al. 2025]



Under-approximation of Cell Bounds [Promies et al. 2025]



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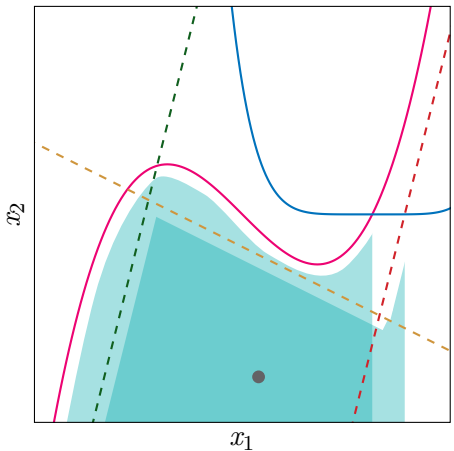

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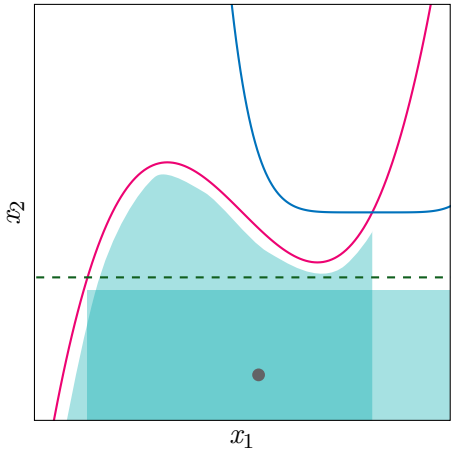

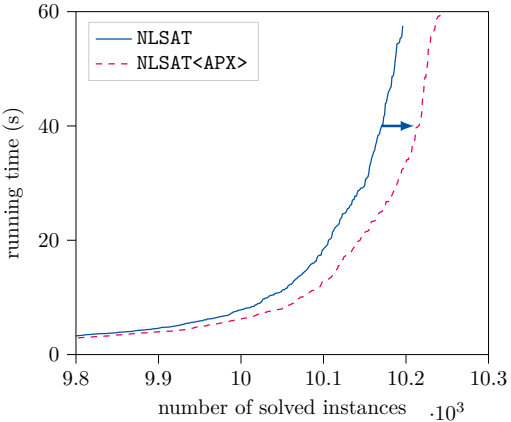

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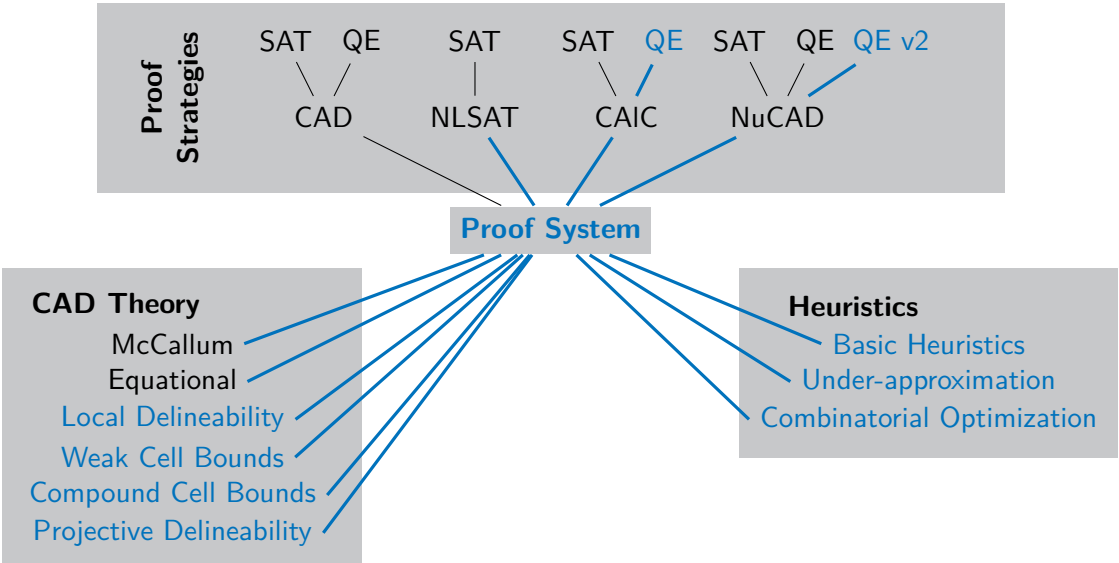

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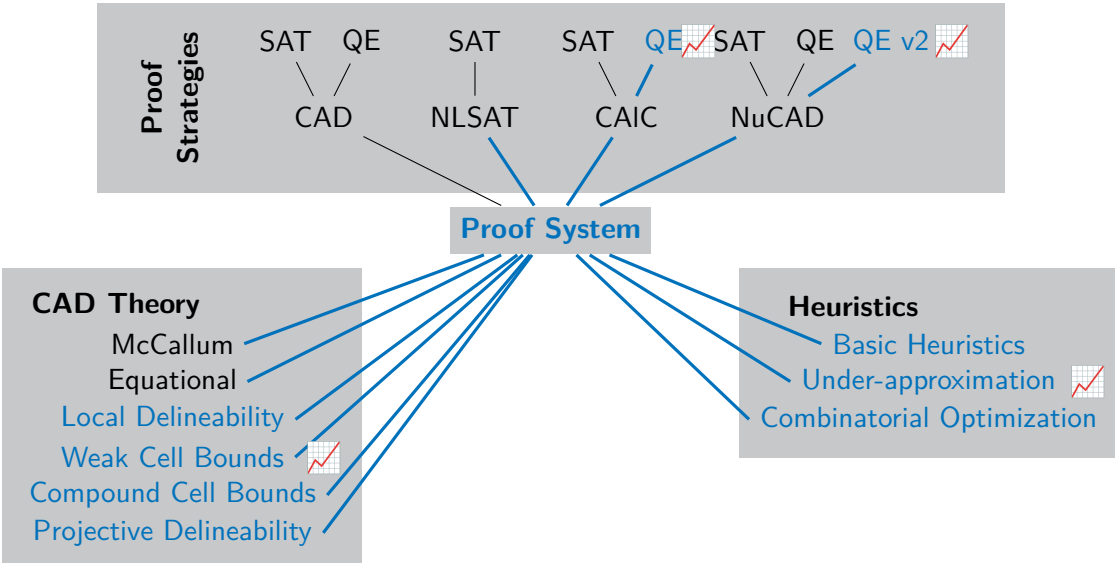

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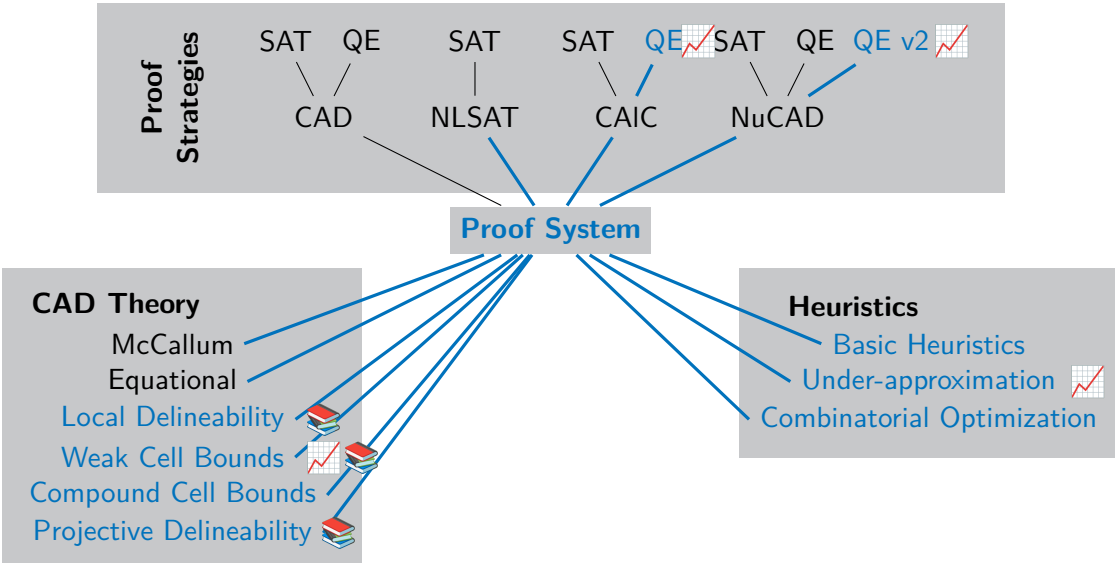

Contributions