

# Exploration-guided Methods for Non-linear Real Arithmetic

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RWTH Aachen University

Dagstuhl Seminar 26091: Revisiting the Foundations of Deduction in a New World  
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# Real Algebra

► Truth of sentences

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- ▶ Quantifier elimination

$$\exists x_2. \exists x_3. x_1^2 + x_2^2 + x_3^2 - 1 < 0 \wedge (x_1 + x_2 + x_3 > 0 \vee x_2 \geq 0) \equiv -1 < x < 1$$

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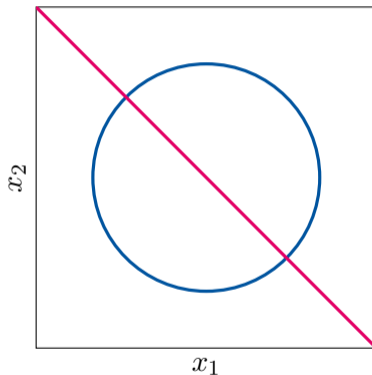
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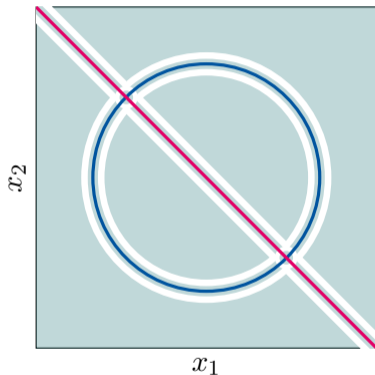
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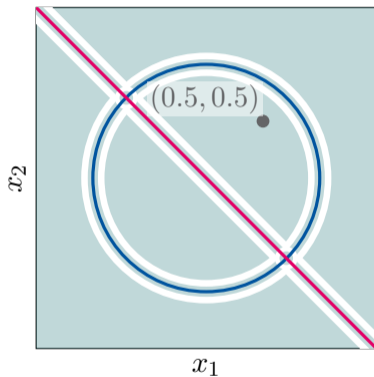
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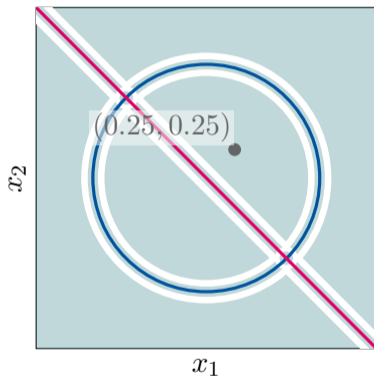
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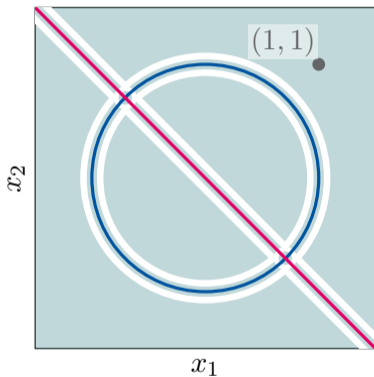
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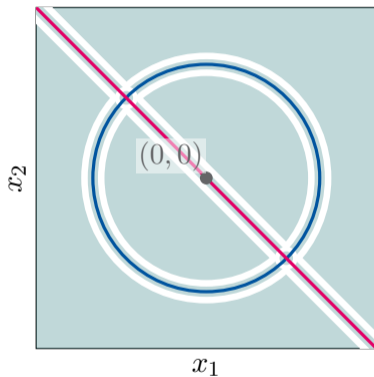
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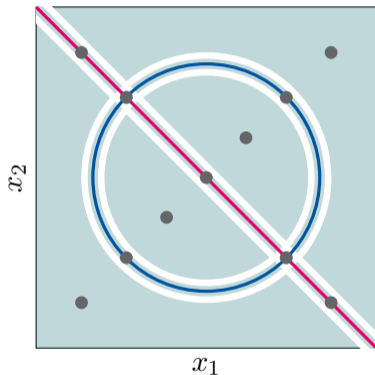
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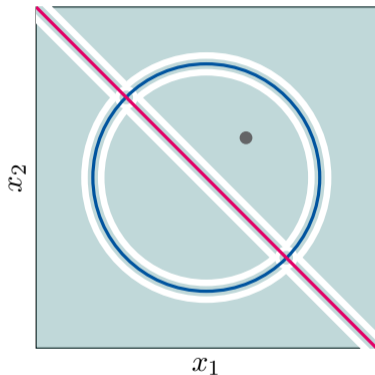
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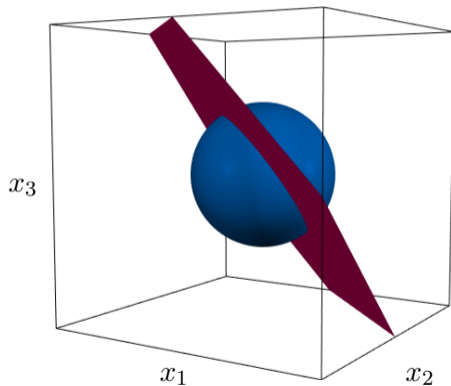
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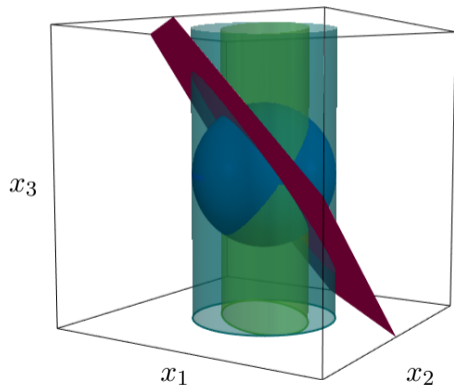
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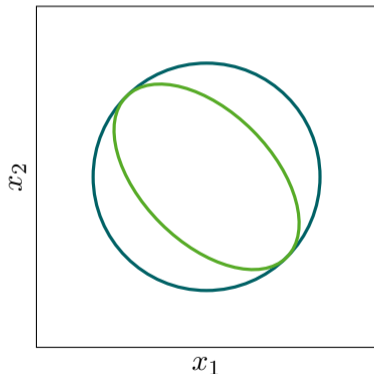


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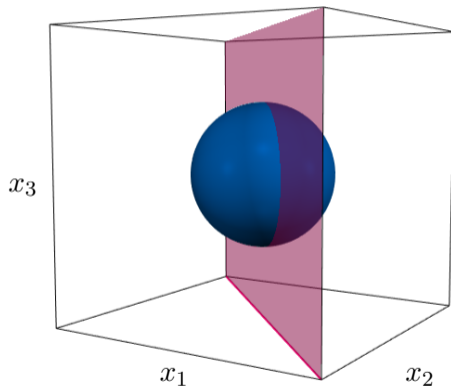


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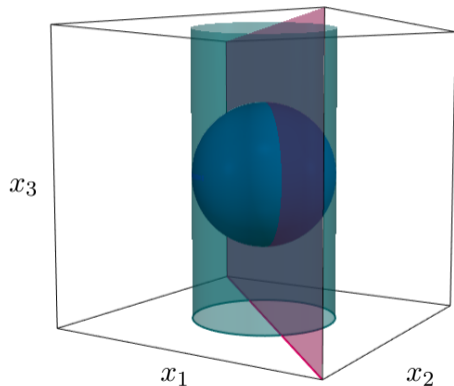
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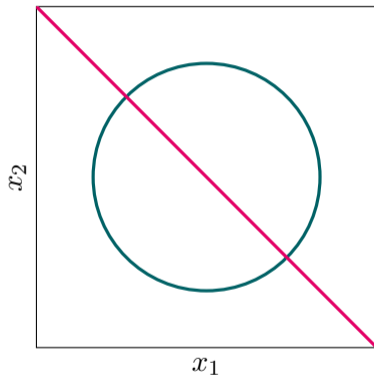


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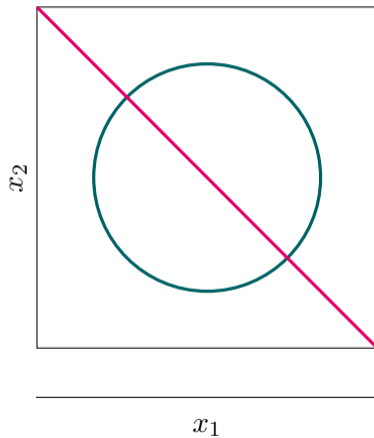


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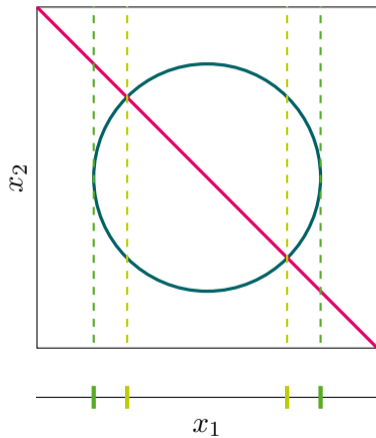
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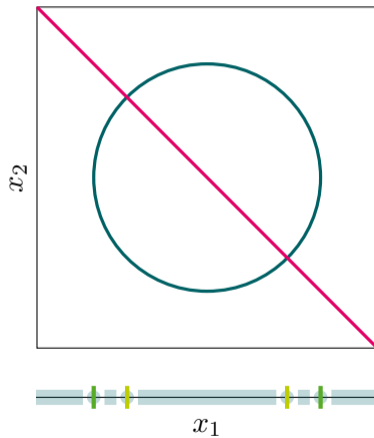
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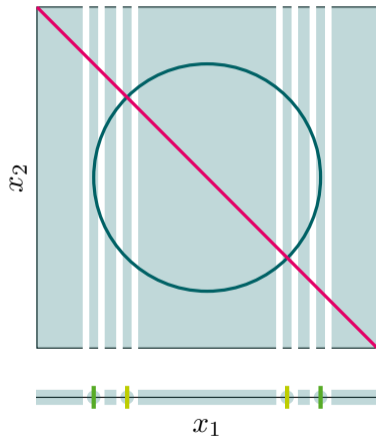
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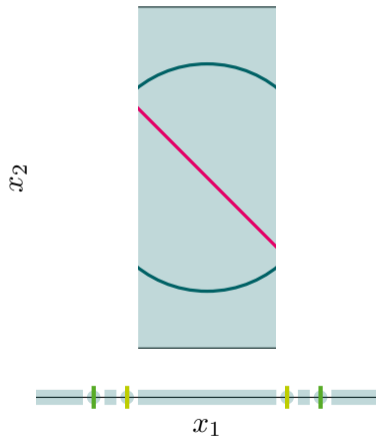
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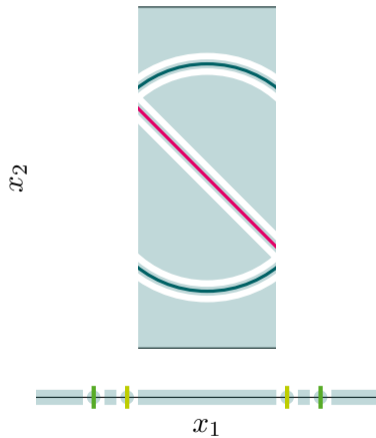
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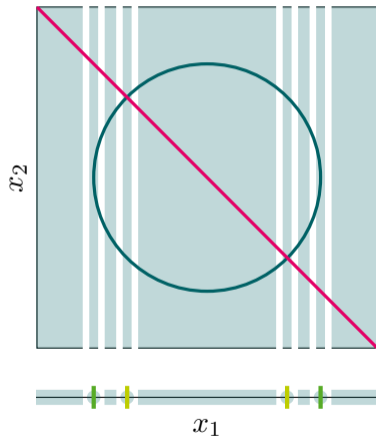
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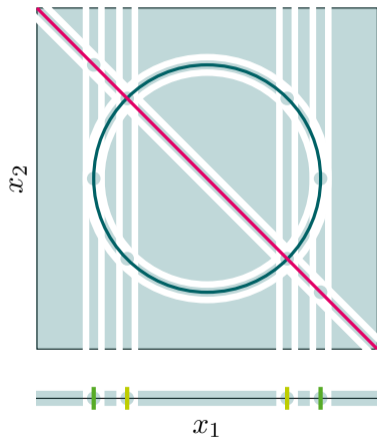
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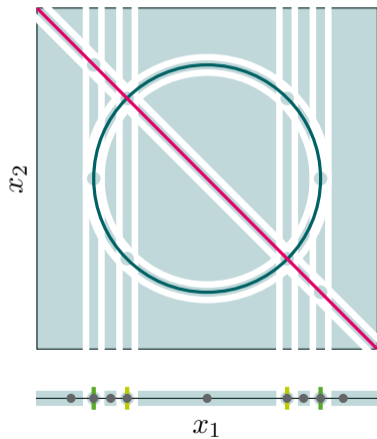
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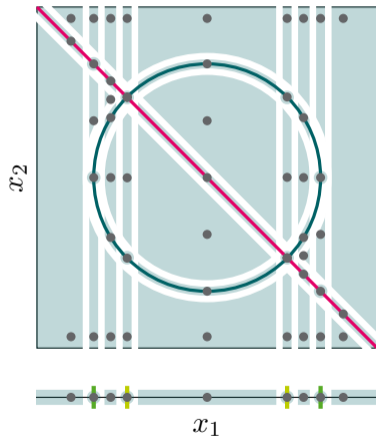
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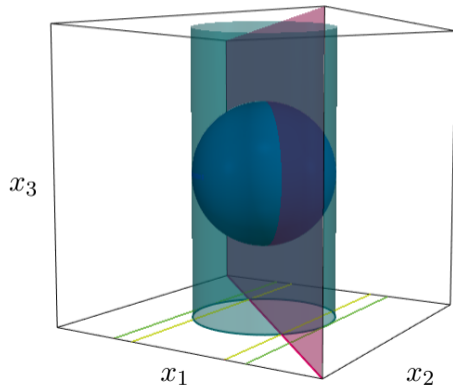
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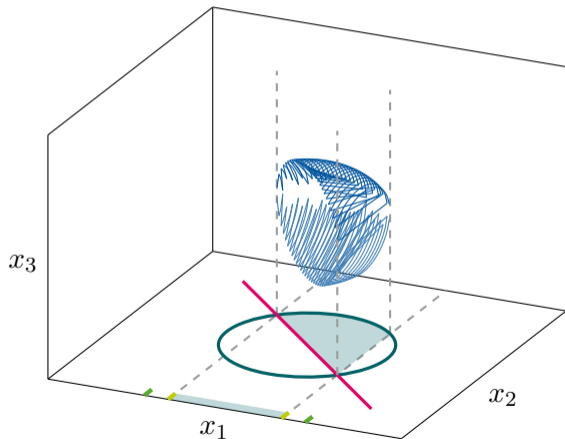
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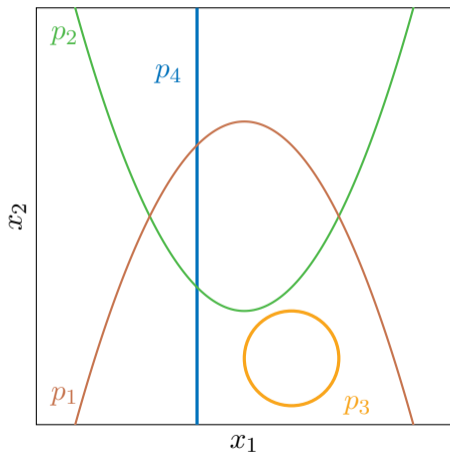
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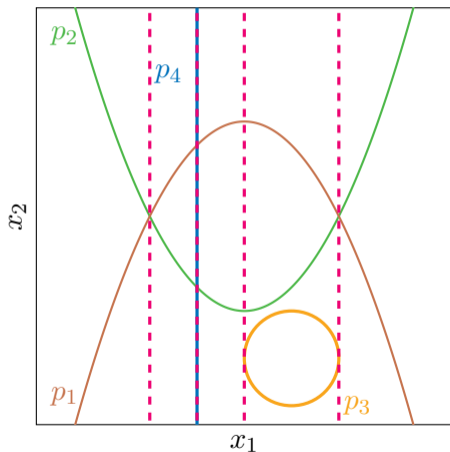
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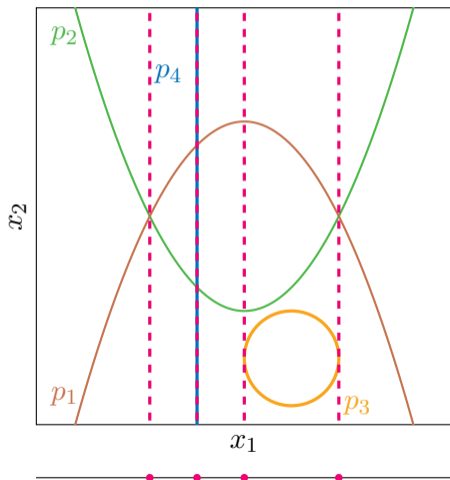
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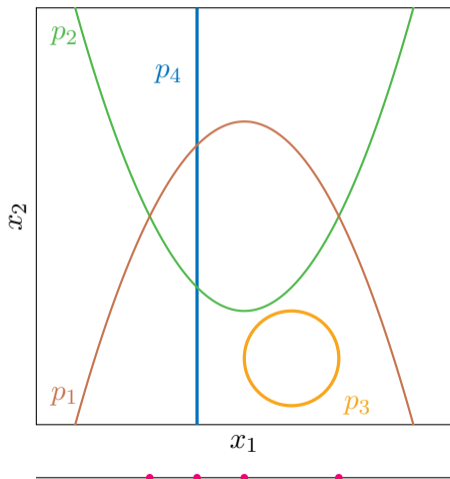
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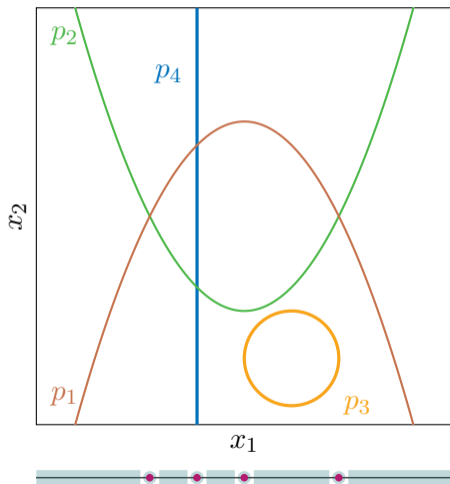
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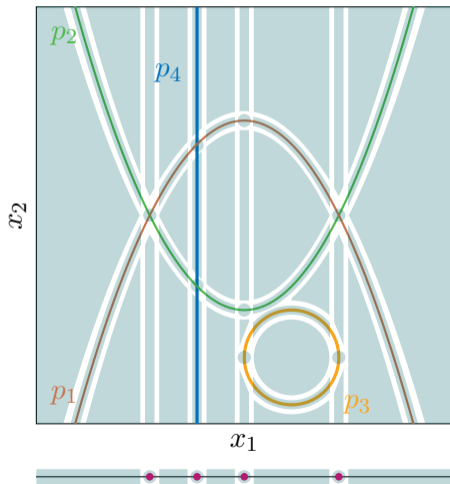
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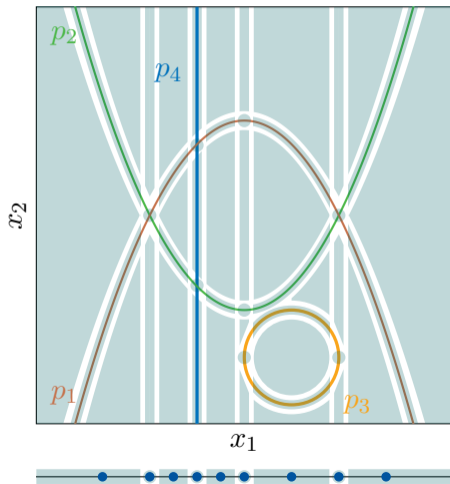
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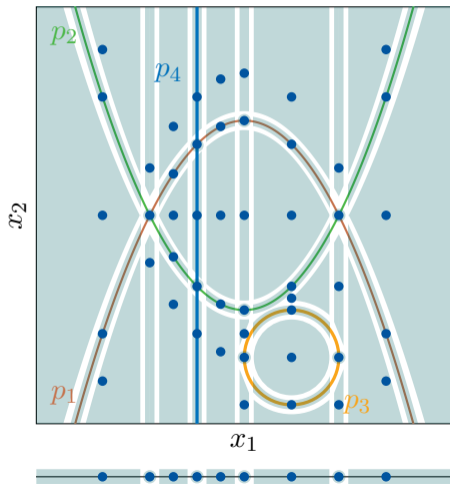
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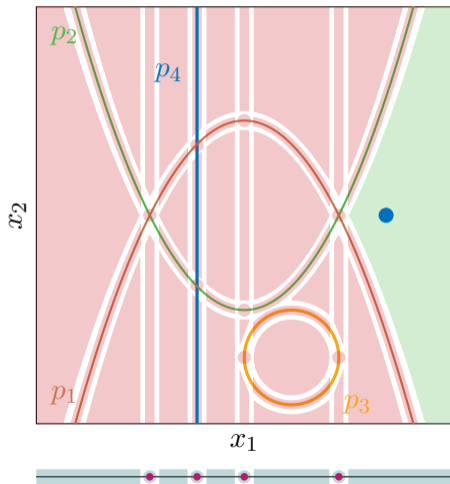
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$2^n$



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Image: Terezo del Río

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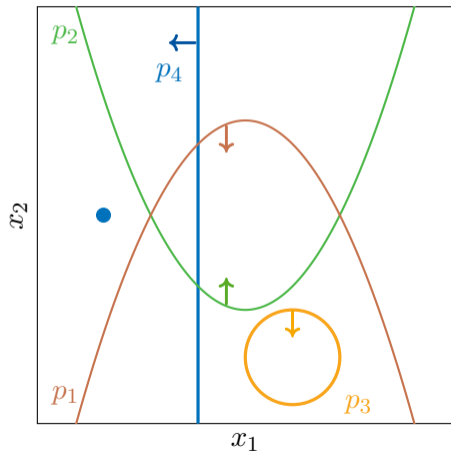
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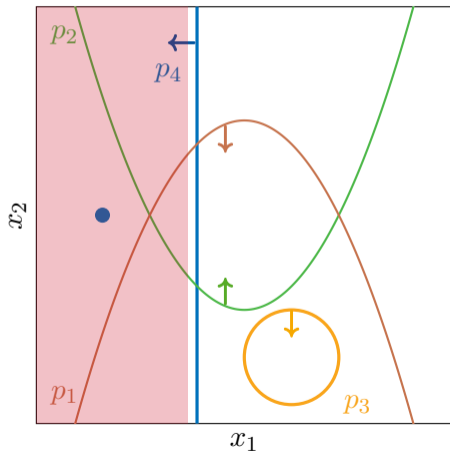
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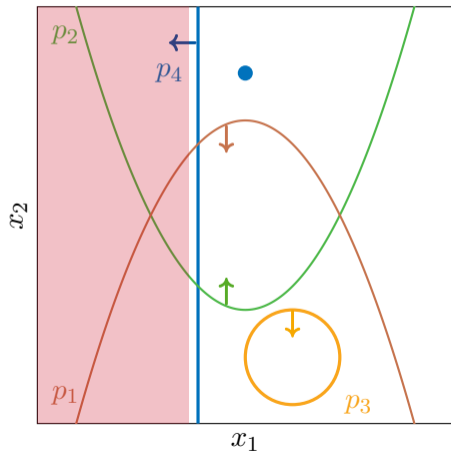
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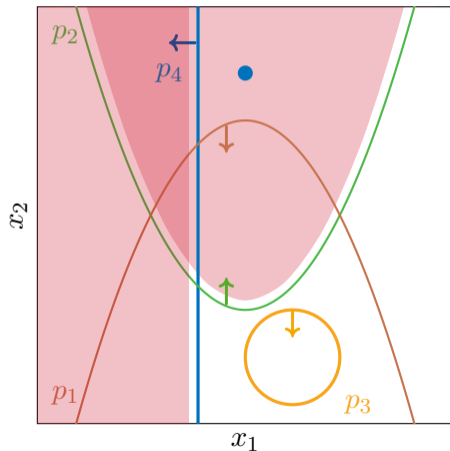
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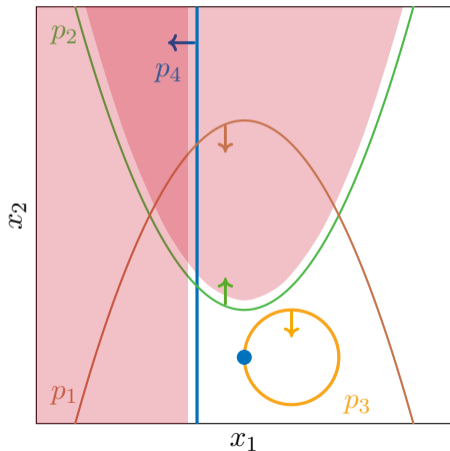
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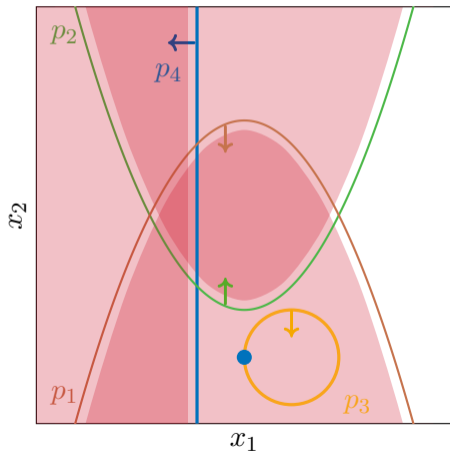
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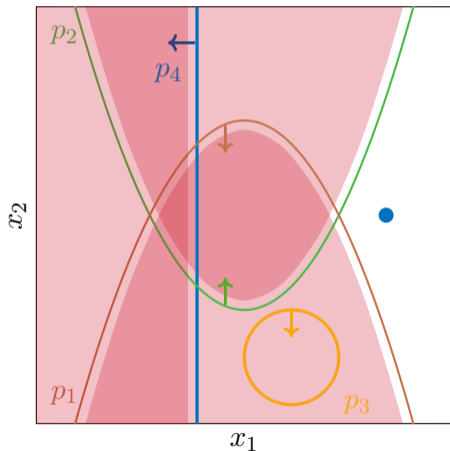
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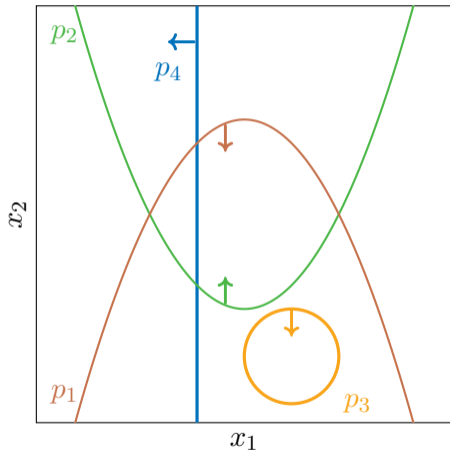
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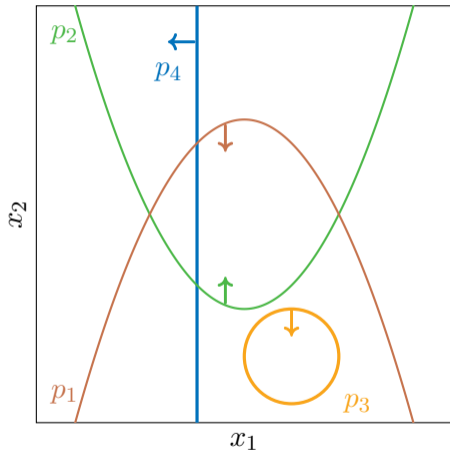
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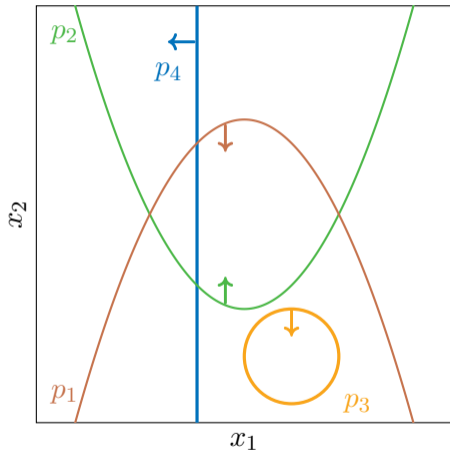
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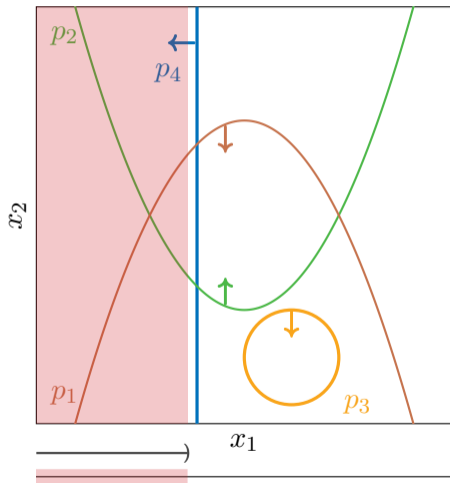
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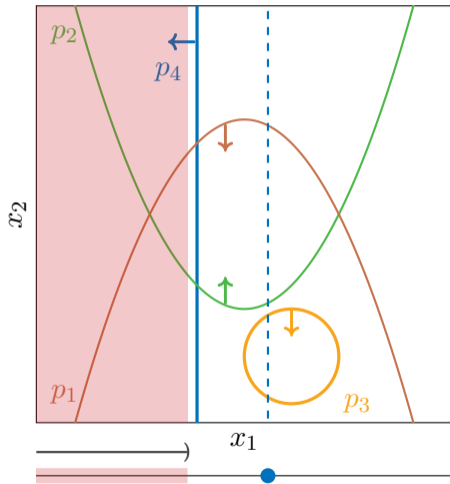
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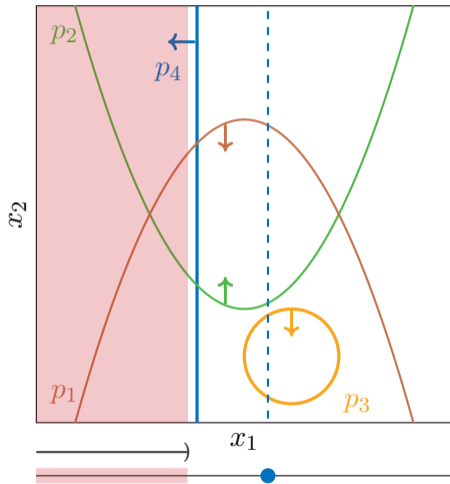
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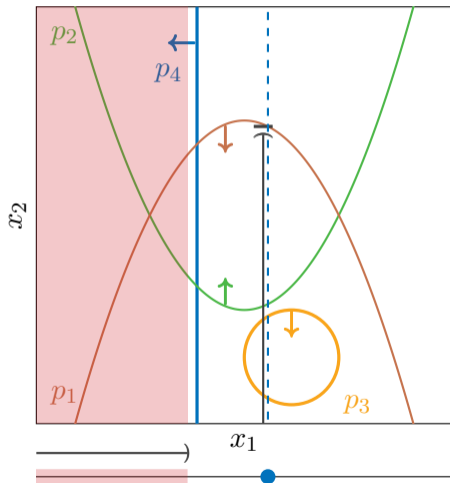
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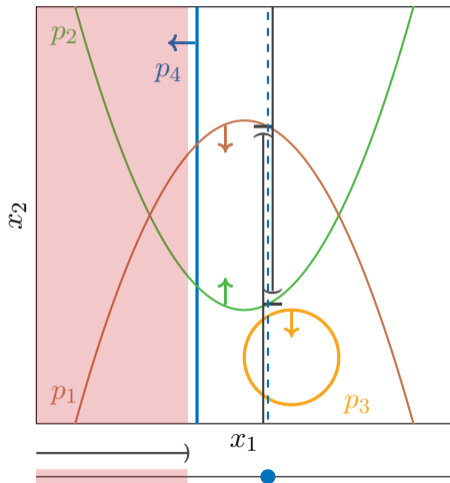
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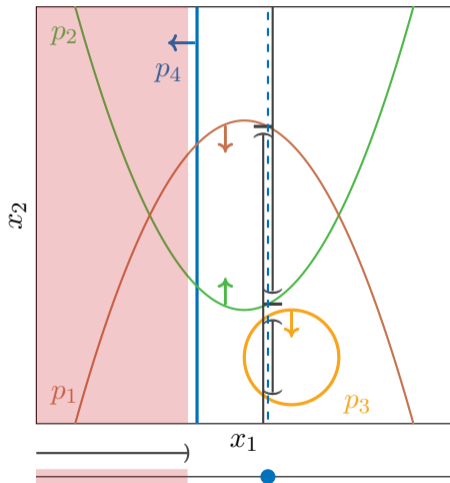
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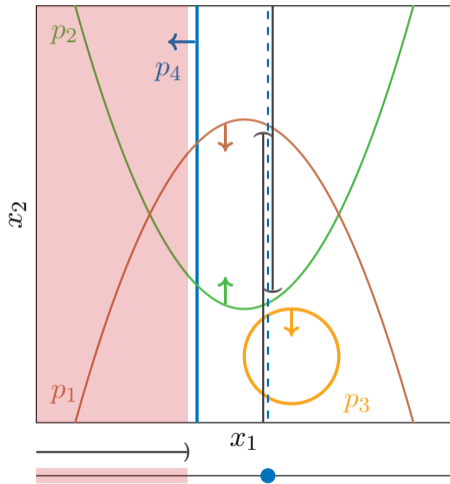
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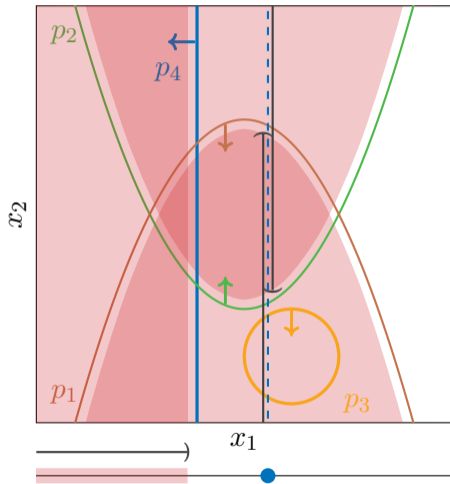
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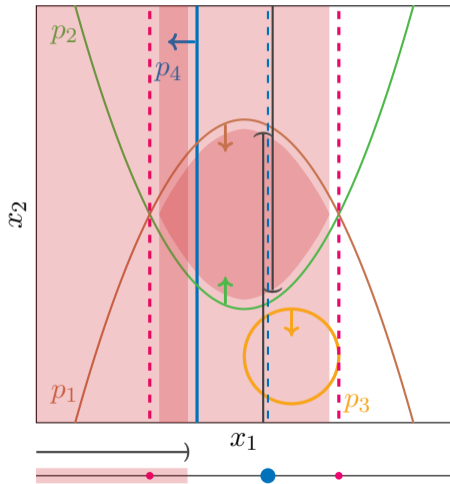
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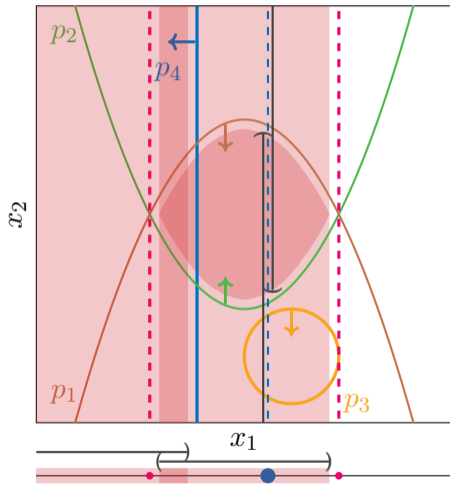
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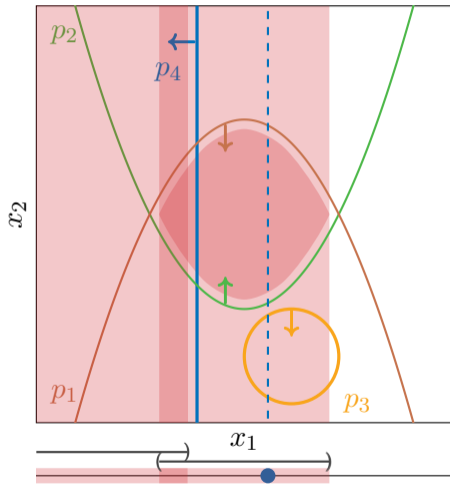
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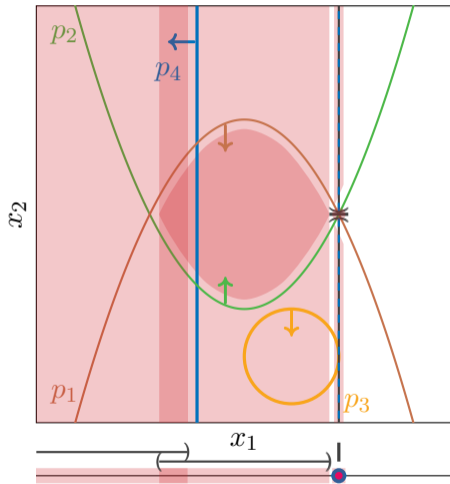
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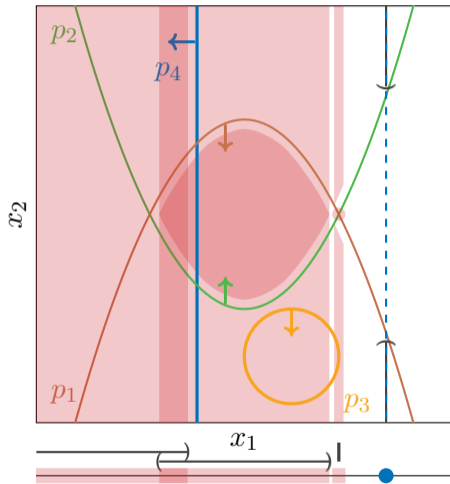
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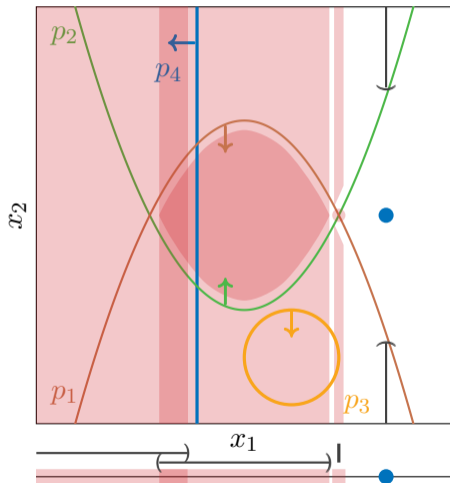
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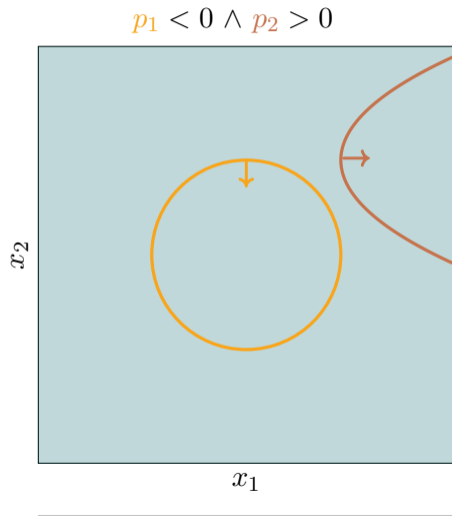


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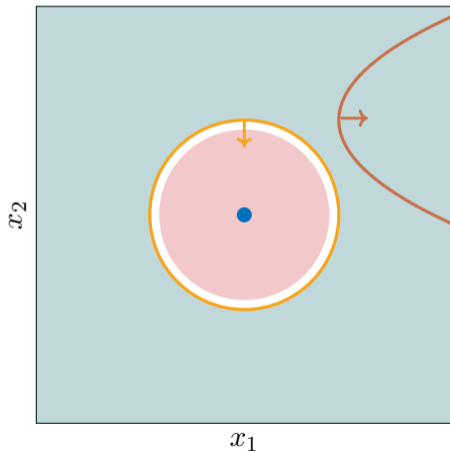


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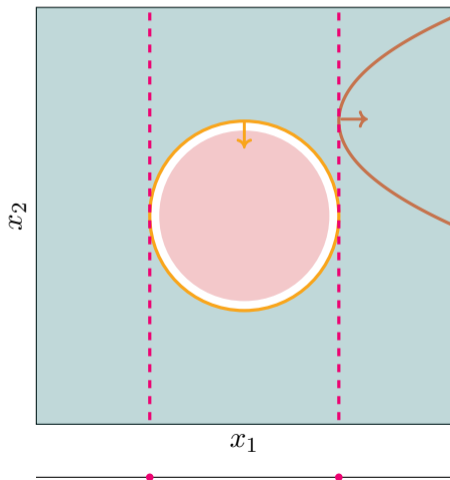
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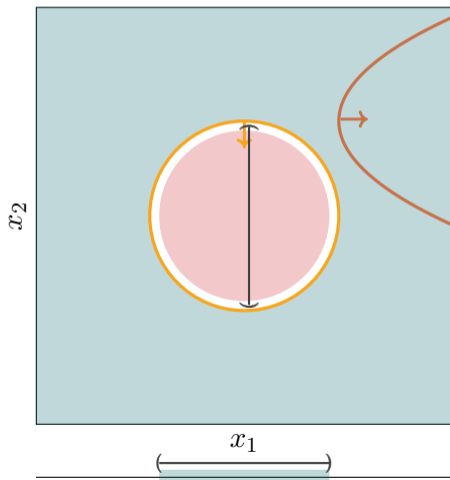
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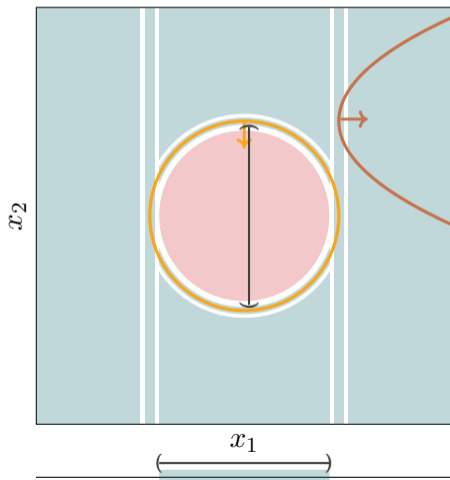
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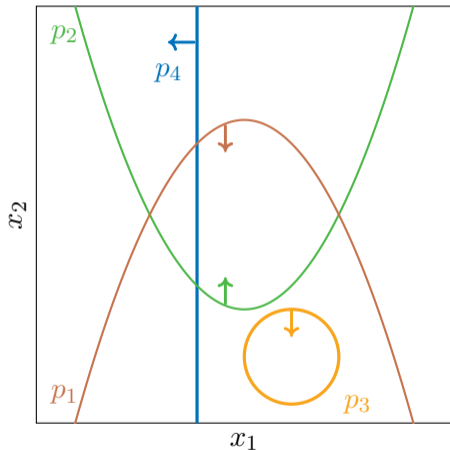
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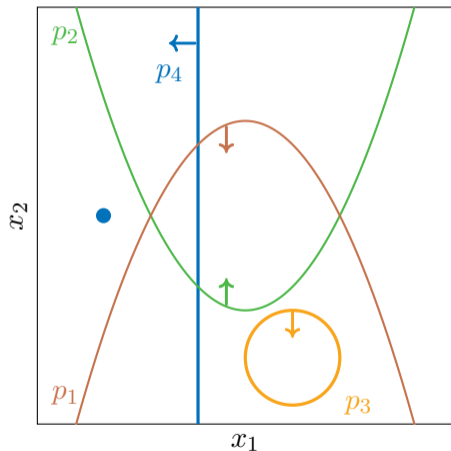
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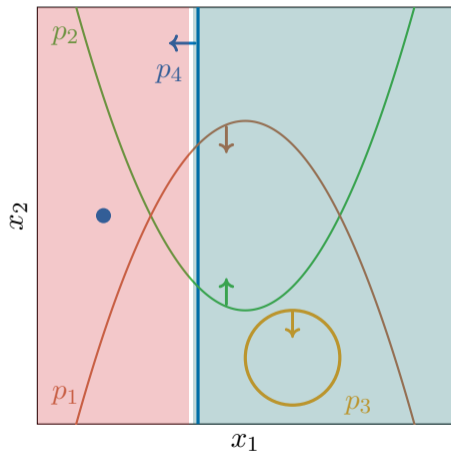
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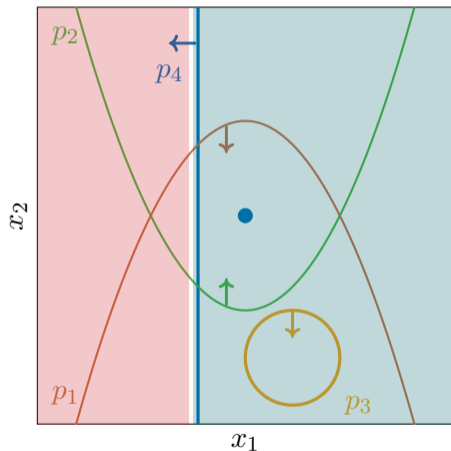
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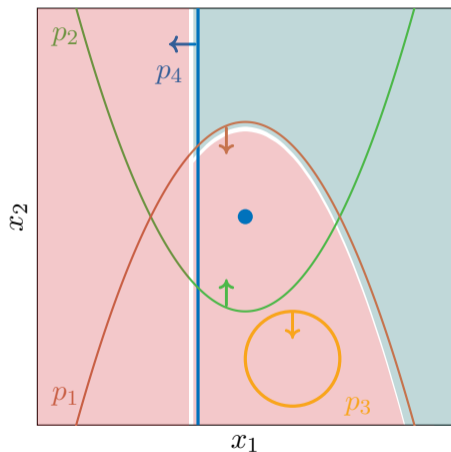
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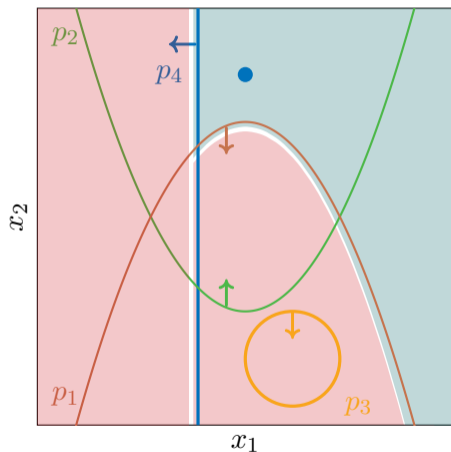
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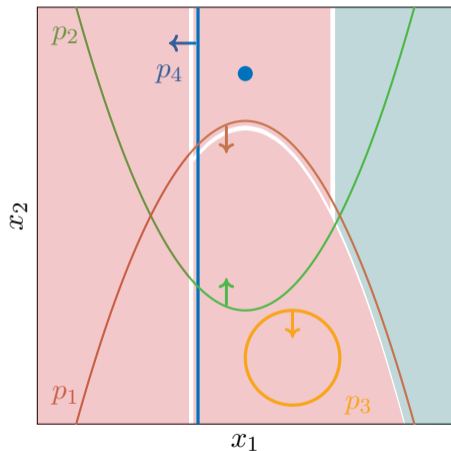
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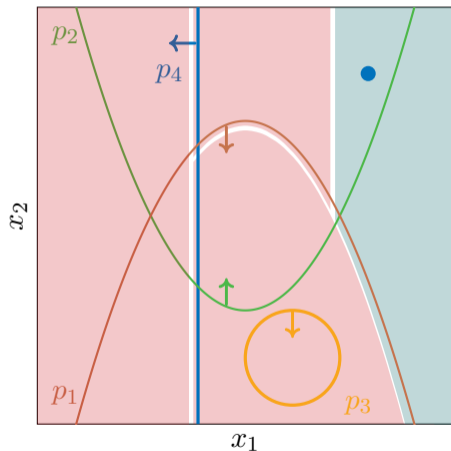
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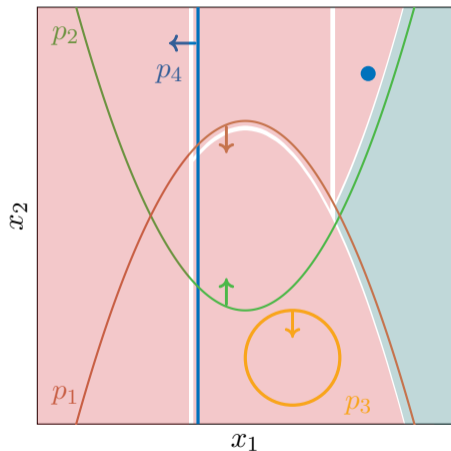
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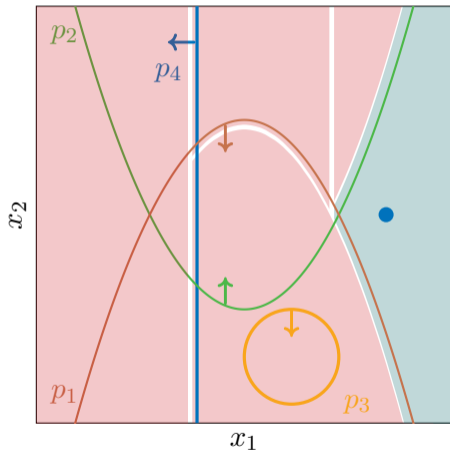
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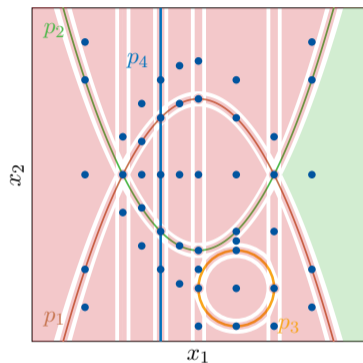


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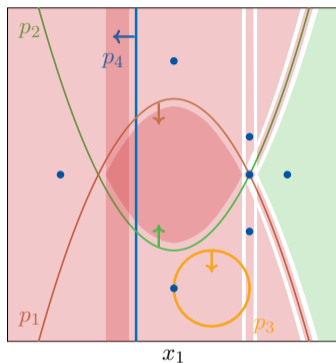
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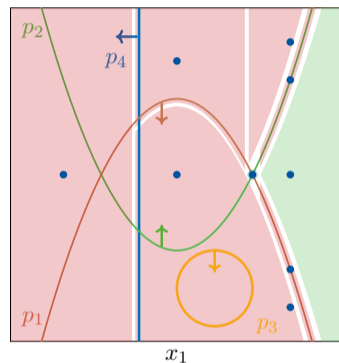
# Exploration-guided Algorithms



Cylindrical Algebraic Decomposition (CAD) [Collins 1975]



NLSAT [Jovanović, de Moura 2012], Cylindrical Algebraic Covering (CAIC) [Ábrahám et al. 2021]

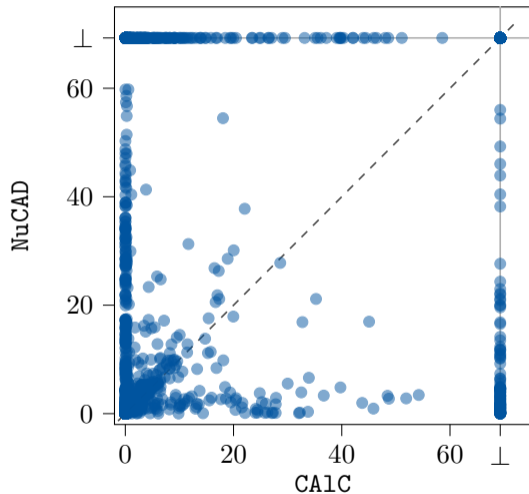


Non-uniform CAD (NuCAD) [Brown 2015]

# Extensions

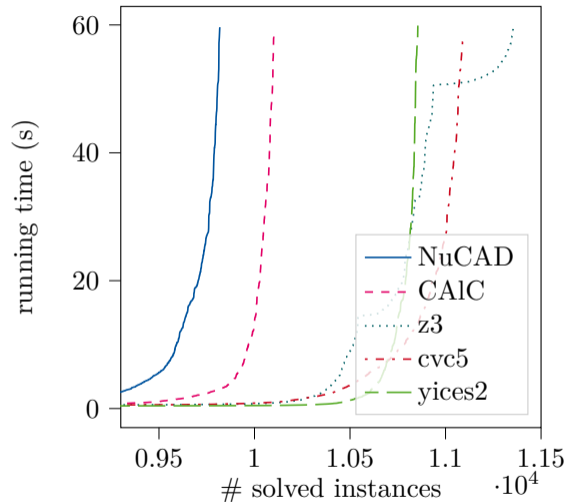
- ▶ Boolean reasoning
- ▶ Quantifier elimination
- ▶ Tweaks of projection

# Comparison of CA1C and NuCAD

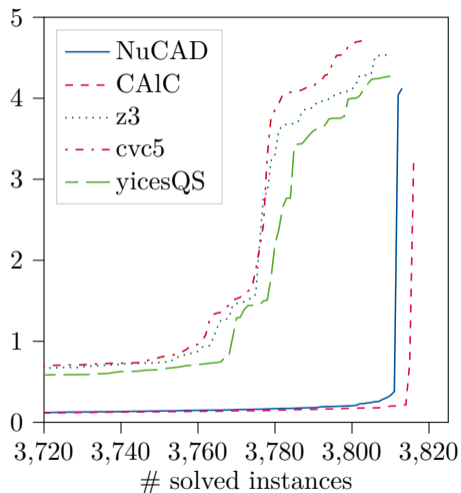


SMT-LIB QF\_NRA

# Comparison with SMT tools

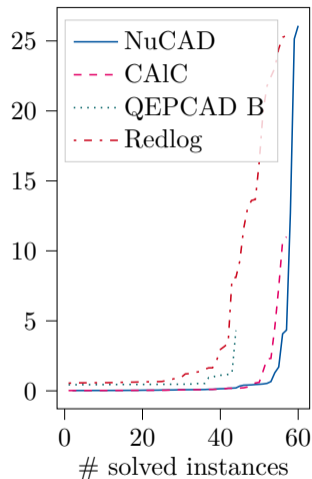


SMT-LIB QF\_NRA



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# Comparison with QE tools



Wilson / Bath benchmark set

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